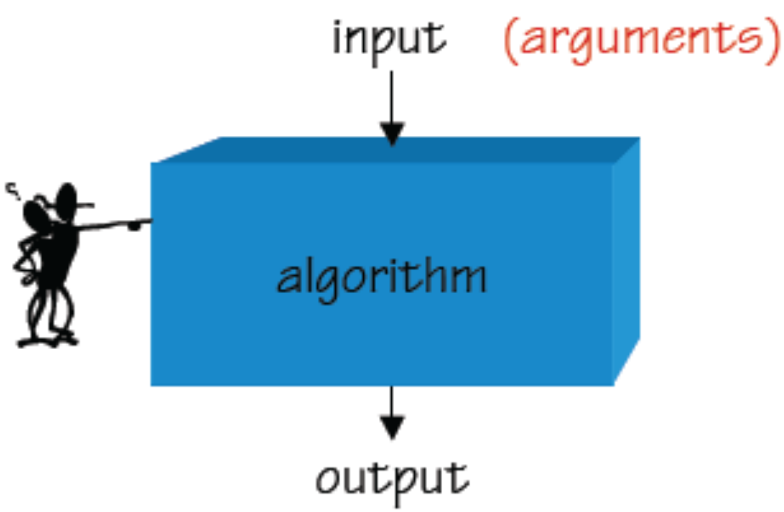
What is an Algorithm?

- An algorithm is a sequence of instructions that one must perform in order to solve a well formulated problem.



Problem: Complexity

Algorithm: Correctness
Complexity

Algorithm vs. Program

- An algorithm is an "abstract" description of a process that is precise, yet general
  - Algorithms are described as generally as possible, so they can be analyzed and proven correct
- Programs are often specific implementations of an algorithm
  - For a specific machine
  - In a specific language

An Example: Buying a CD

1. Go to Best Buy
2. Go to the correct music genre section
3. Search the racks for the artist's name
4. Take a copy of the CD.
5. Go to the register.
6. Check out using credit card.
7. Rip it onto your laptop.

1. Sign into iTunes.com
2. Go to iTunes Store
3. Type CD title into search
4. Scroll through Album list to find CD cover
5. Click "Buy Album".
6. Accept Credit Card charge
7. Agree to download

Two Observations

- Given a problem, there may be more than one correct algorithms.
- However, the costs to perform different algorithms may be different.
- We can measure costs in several ways
  - In terms of time
  - In terms of space

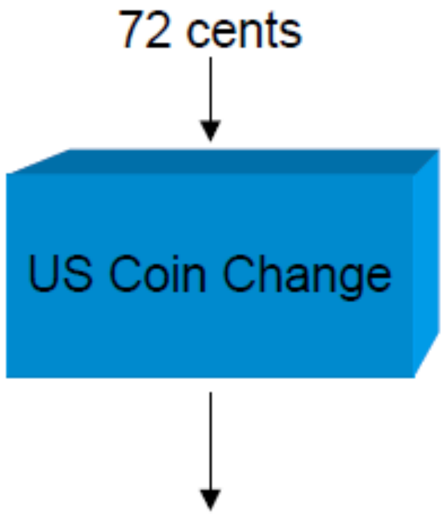
Correctness

- An algorithm is correct only if it produces correct result for all input instances.

– If the algorithm gives an incorrect answer for one or more input instances, it is an incorrect algorithm.

- Coin change problem
  – Input: an amount of money *M in cents*
  – Output: the smallest number of coins
- US coin change problem

US Coin Change



Two quarters, 22 cents left

Two dimes, 2 cents left

Two pennies

$r \leftarrow Amount$

$q \leftarrow r/25$

$r \leftarrow r - 25 \cdot q$

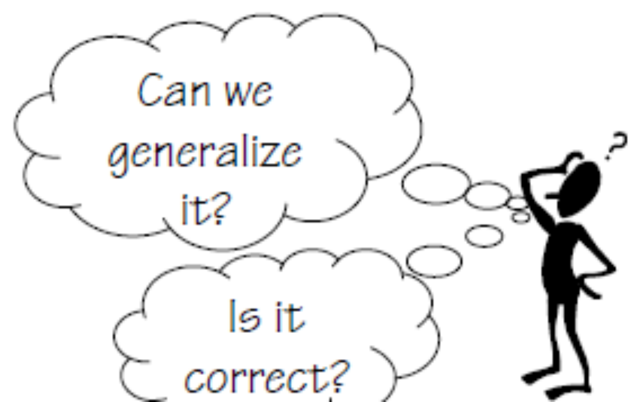
$d \leftarrow r/10$

$r \leftarrow r - 10 \cdot d$

$n \leftarrow r/5$

$r \leftarrow r - 5 \cdot n$

$p \leftarrow r$



Change Problem

- Input:
  - an amount of money "*Amount*"
  - an array of denominations *c = (c1, c2, …, cd) in decreasing values*
- Output: the smallest number of coins

$Amount = 40$
$c = (25, 20, 10, 5, 1)$

```
r      Amount
n      0
For k    1 to d
    i_k    r / c_k
    n      n + i_k
    r      r # c_k   i_k
return n
```

→ 3

The correct answer should be 2.

Incorrect algorithm!

Is it correct?

Complexity of an Algorithm?

- Complexity — the cost in time and space of an algorithm as a function of the input's size
  - Correct algorithms may have different complexities.
- The cost to perform an instruction may vary dramatically.
  - An instruction may be an algorithm itself.
  - The complexity of an algorithm is NOT equivalent to the number of instructions.
- Thinking algorithmically…

Recursive Algorithms

- ***Recursion*** *is technique for describing an algorithm* in terms of itself.
  - These recursive calls are to simpler, or reduced, versions of the original calls.
  - The simplest versions, called "base cases", are merely declared (because the answer is known).

Recursive definition: factorial(n) = $n \times$ factorial(n -1)

Base case: factorial(1) =1

Example of Recursion

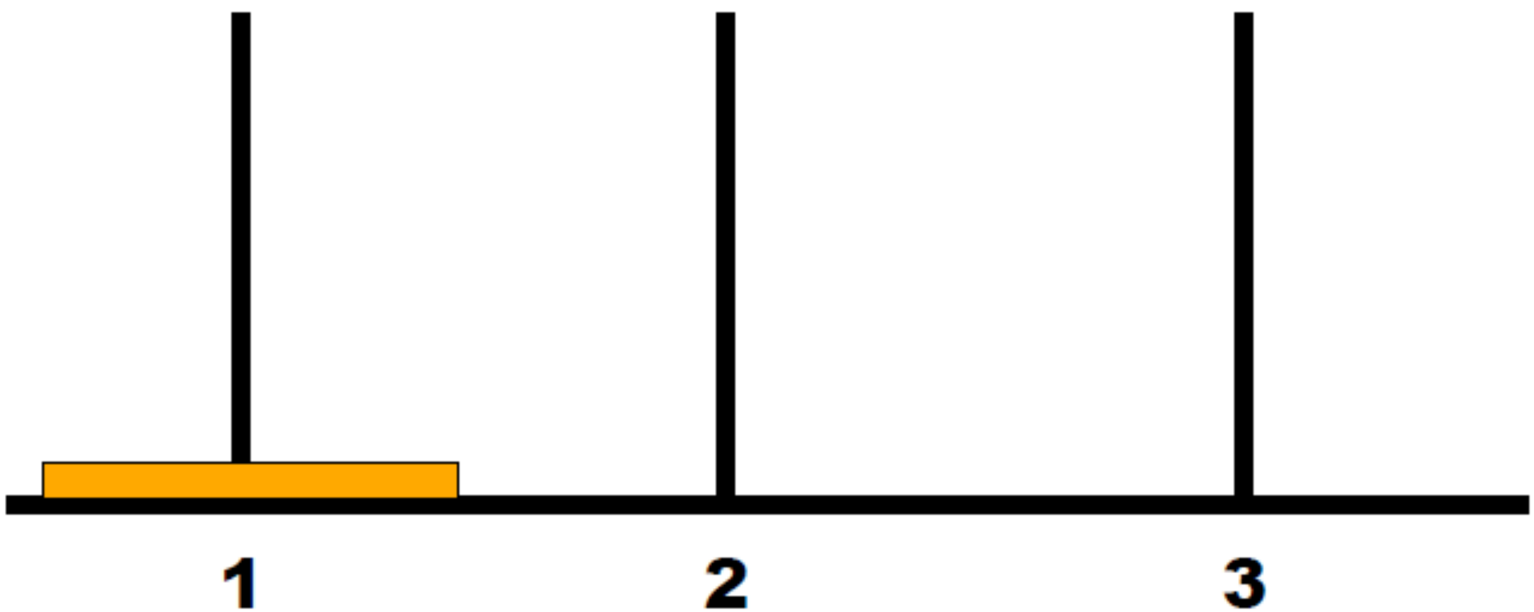
```
def factorial(n):
        if (n == 1):
            return 1
        else:
                return n*factorial(n-1)
```

- Recursion is a useful technique for specifying algorithms concisely
- Recursion can be used to decompose large problems into smaller simpler ones
- Recursion can illuminate the non-obvious

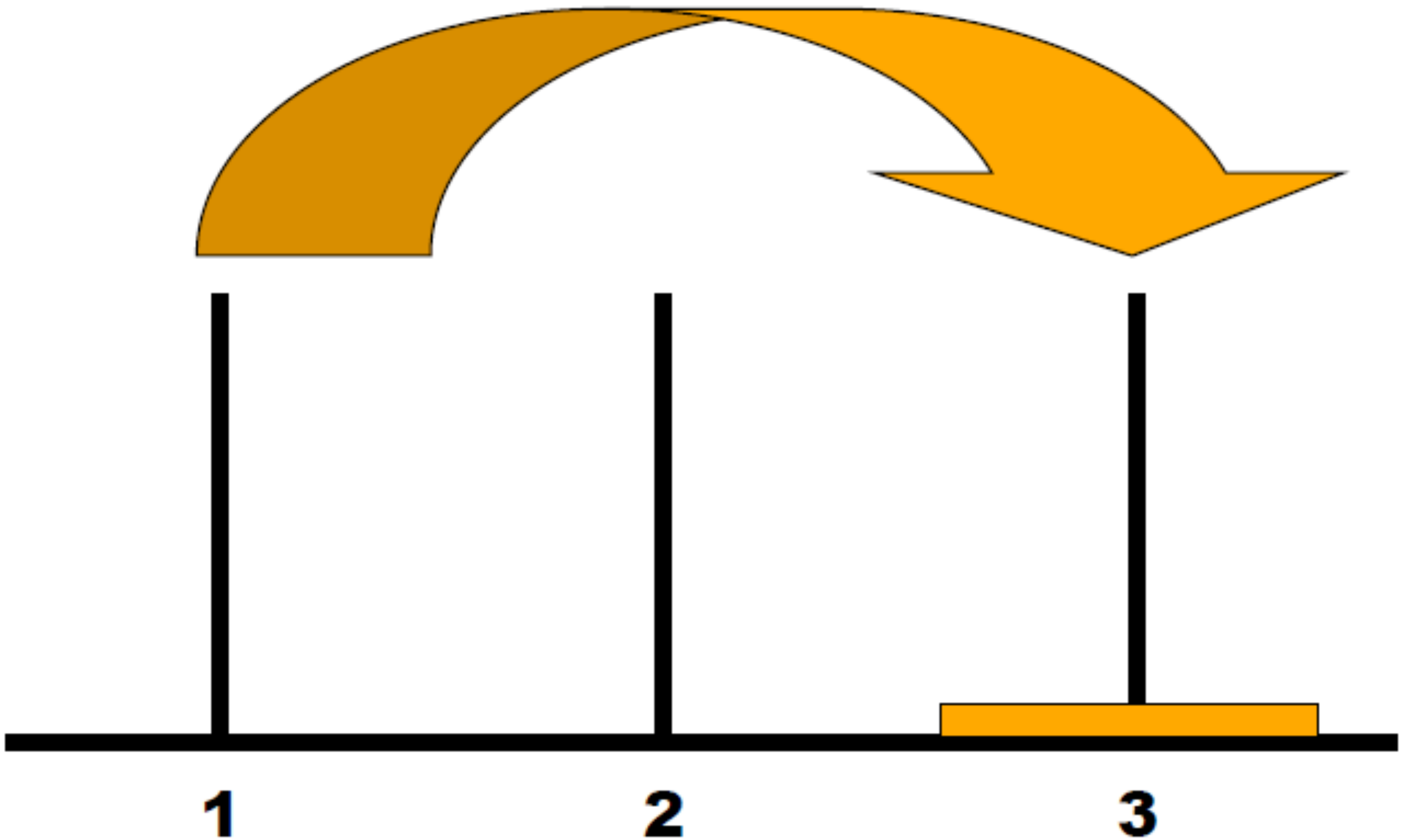
Towers of Hanoi

- There are three pegs and a number of disks with
  decreasing radii (smaller ones on top of larger
  ones) stacked on Peg 1.
- Goal: move all disks to Peg 3.
- Rules:
  - At each move a disk is moved from one peg to another.
  - Only one disk may be moved at a time, and it must be the top disk on a tower.
  - A larger disk may never be placed upon a smaller disk.

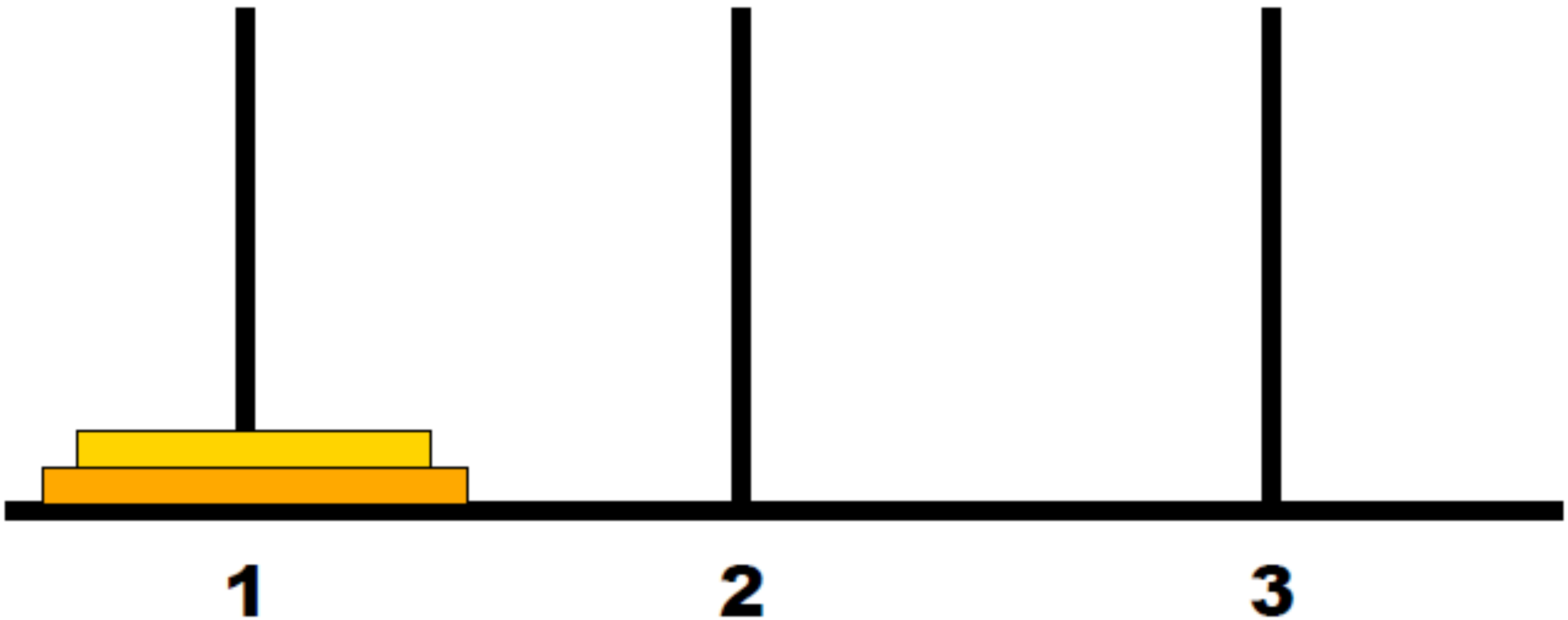
A single disk tower



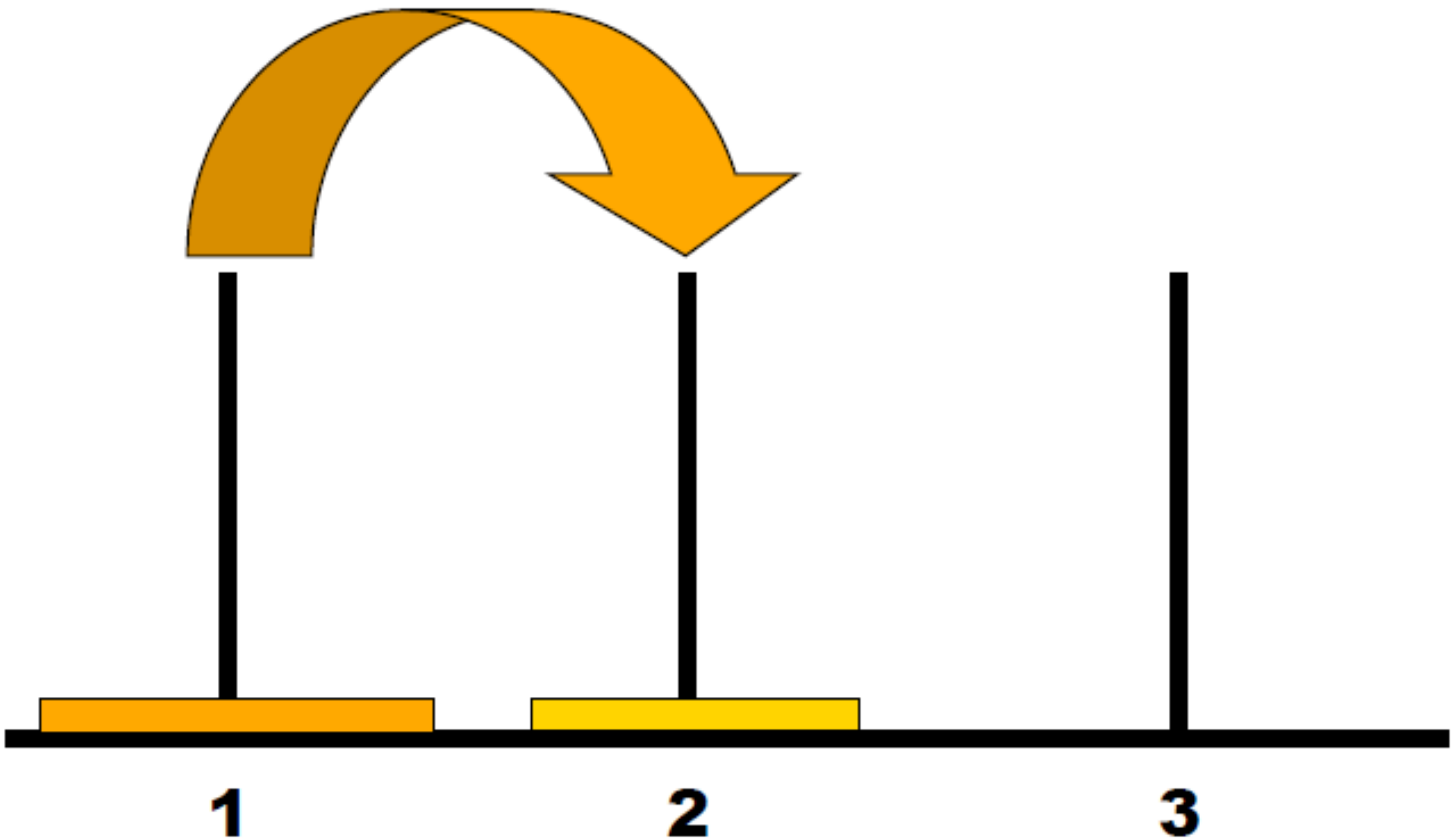
A single disk tower



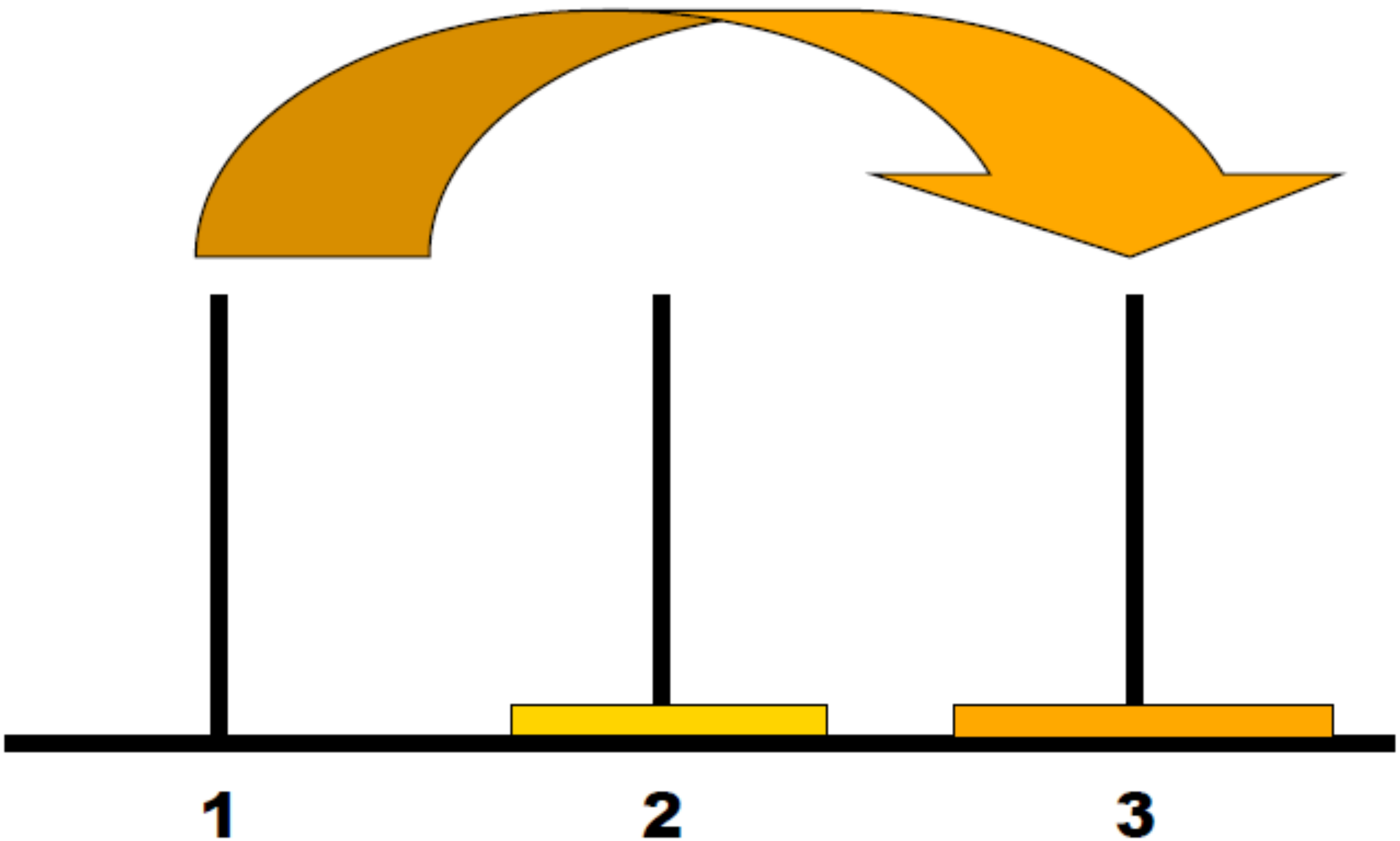
A two disk tower



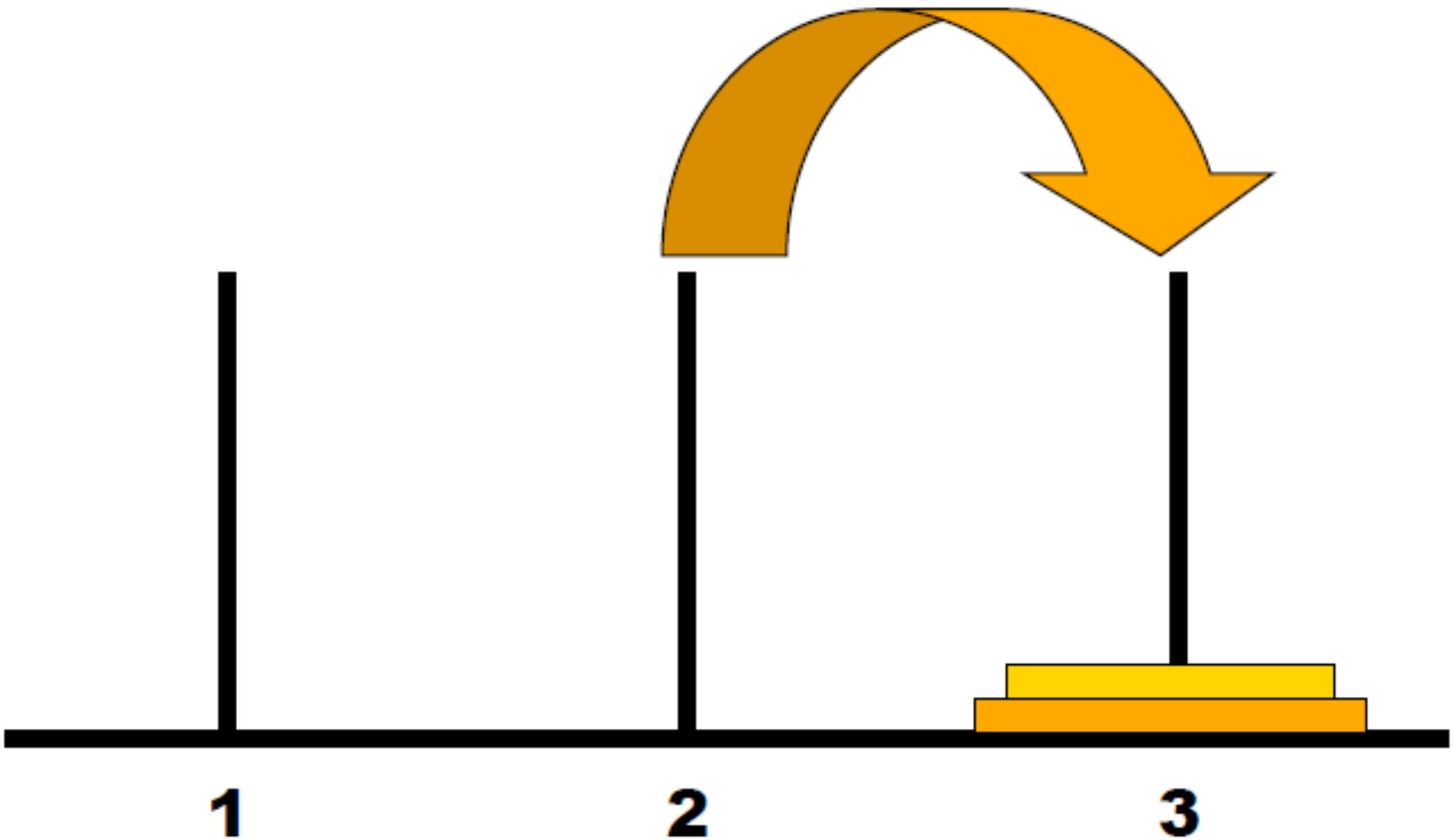
Move 1



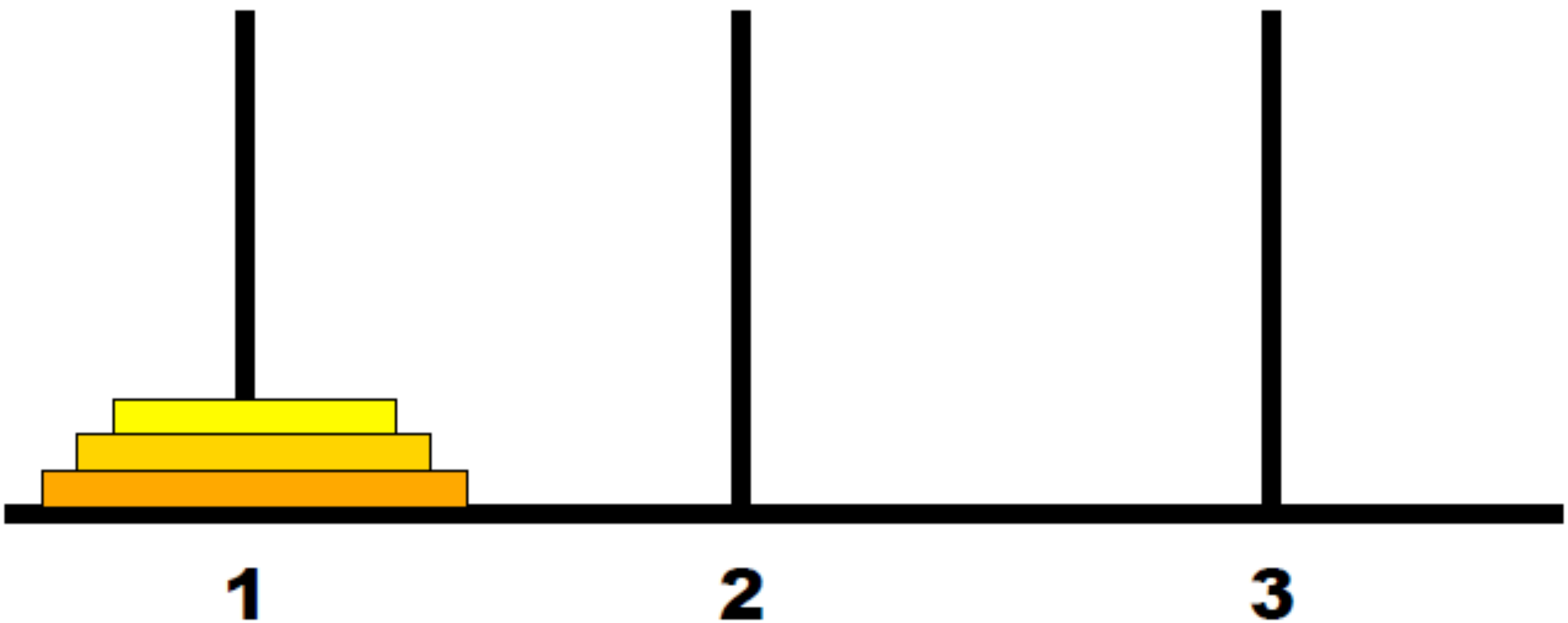
Move 2



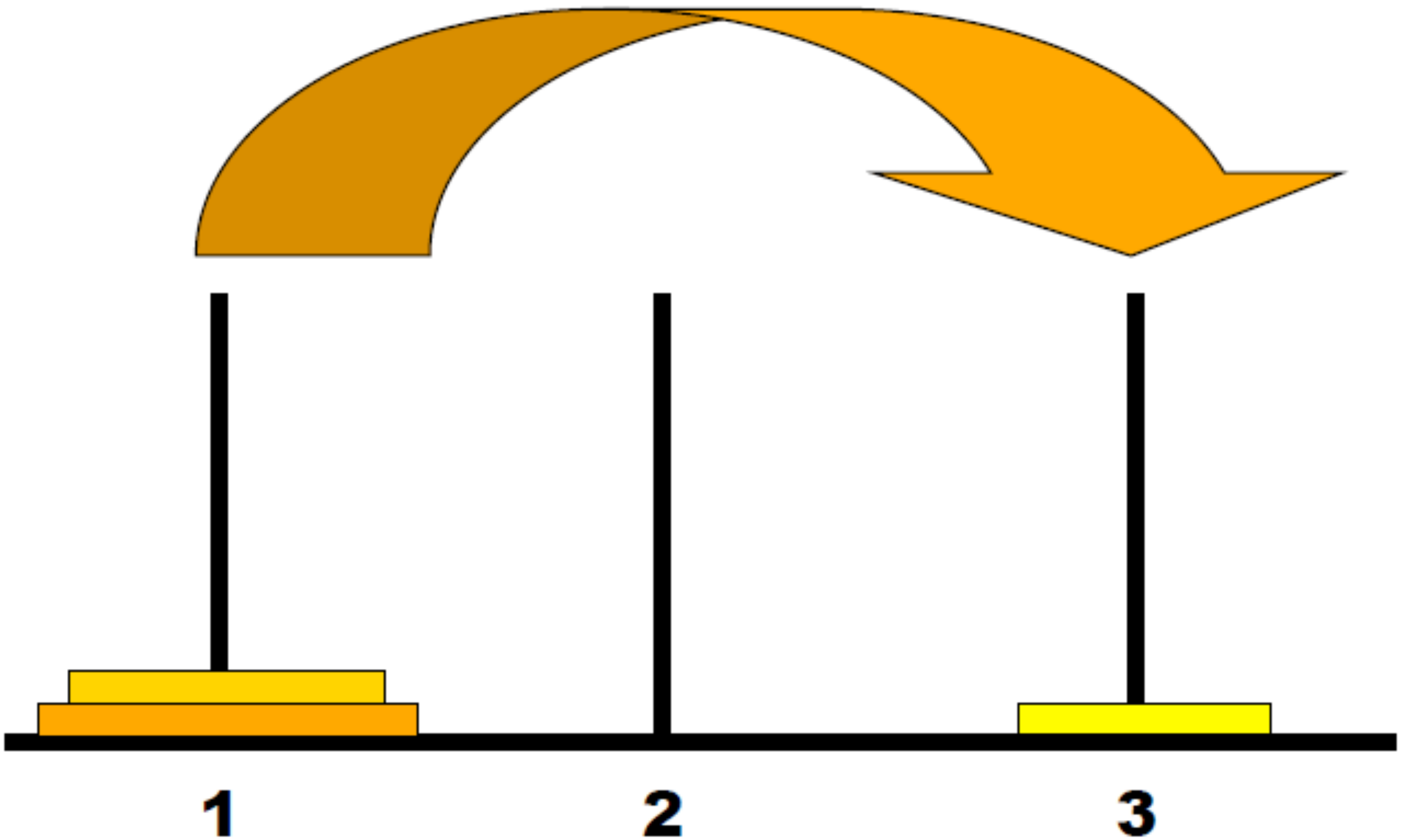
Move 3



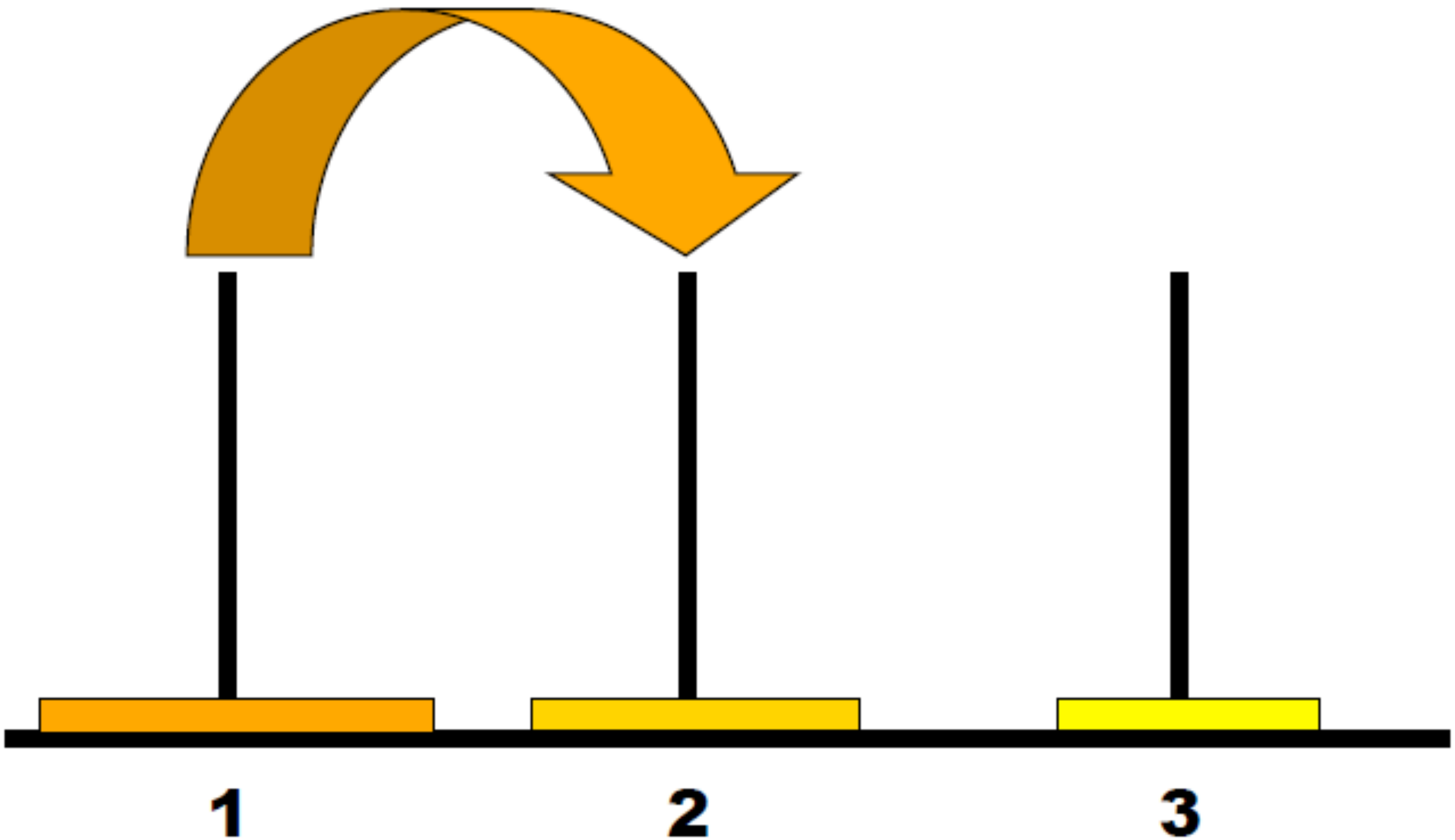
A three disk tower



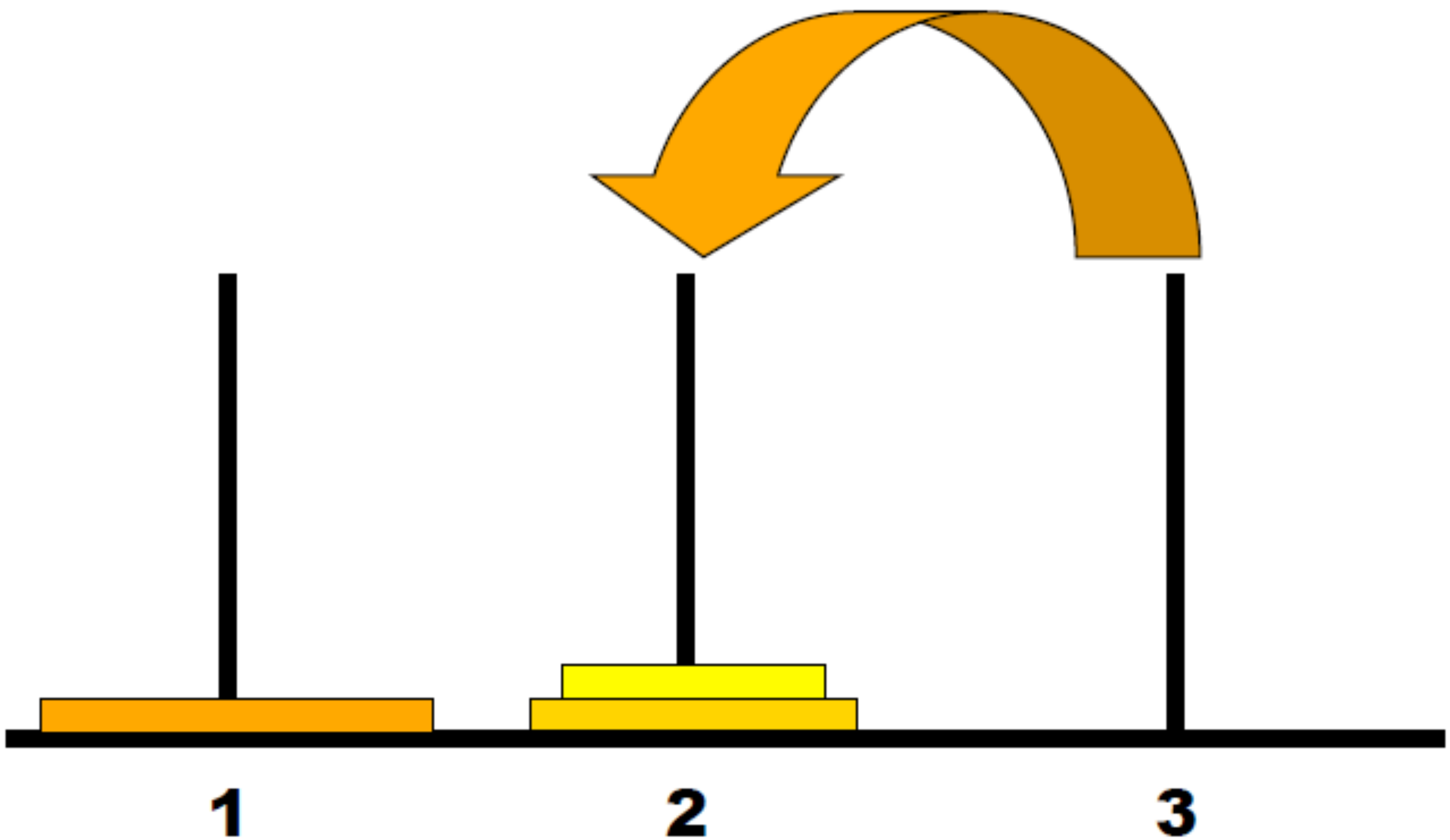
Move 1



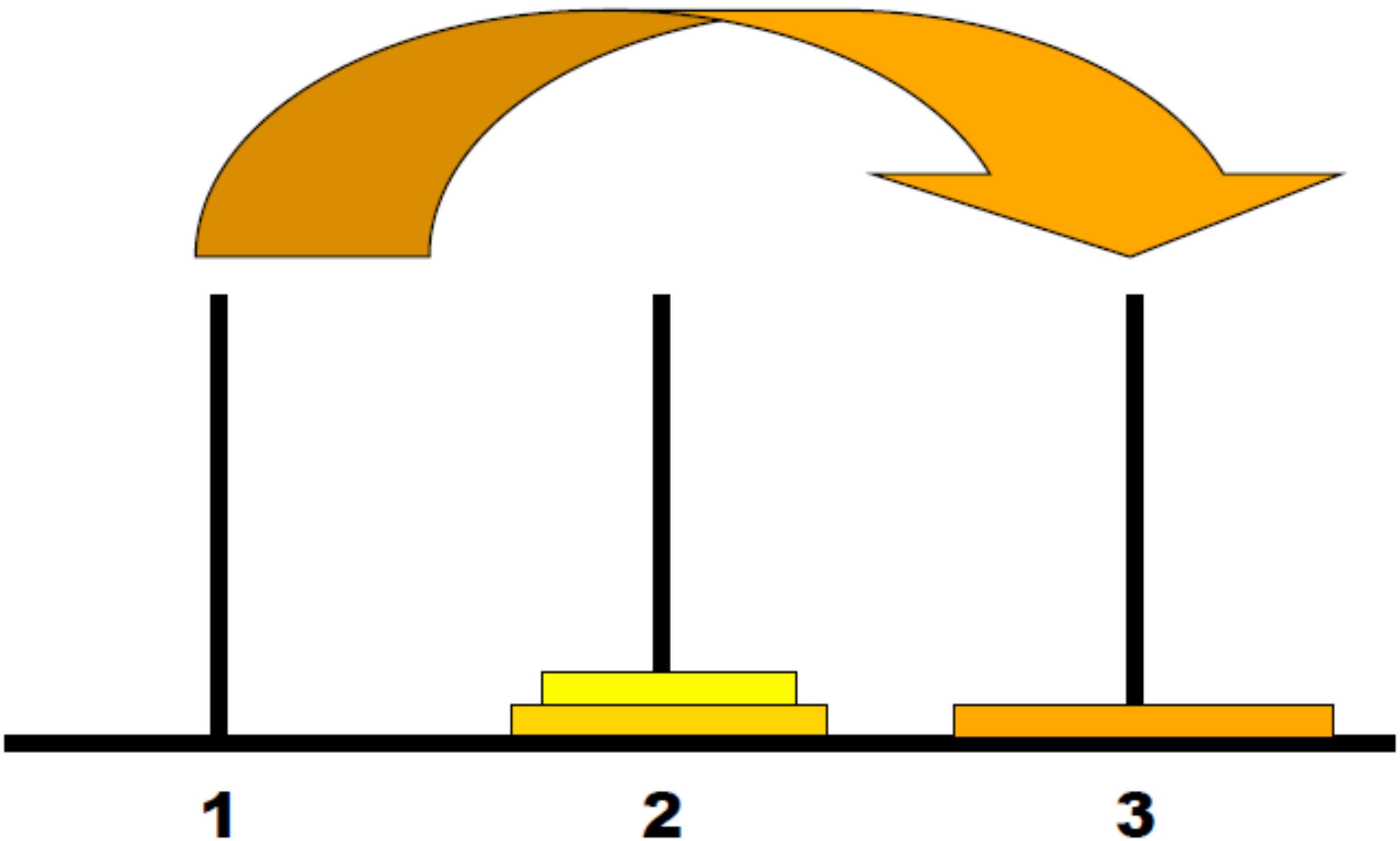
Move 2



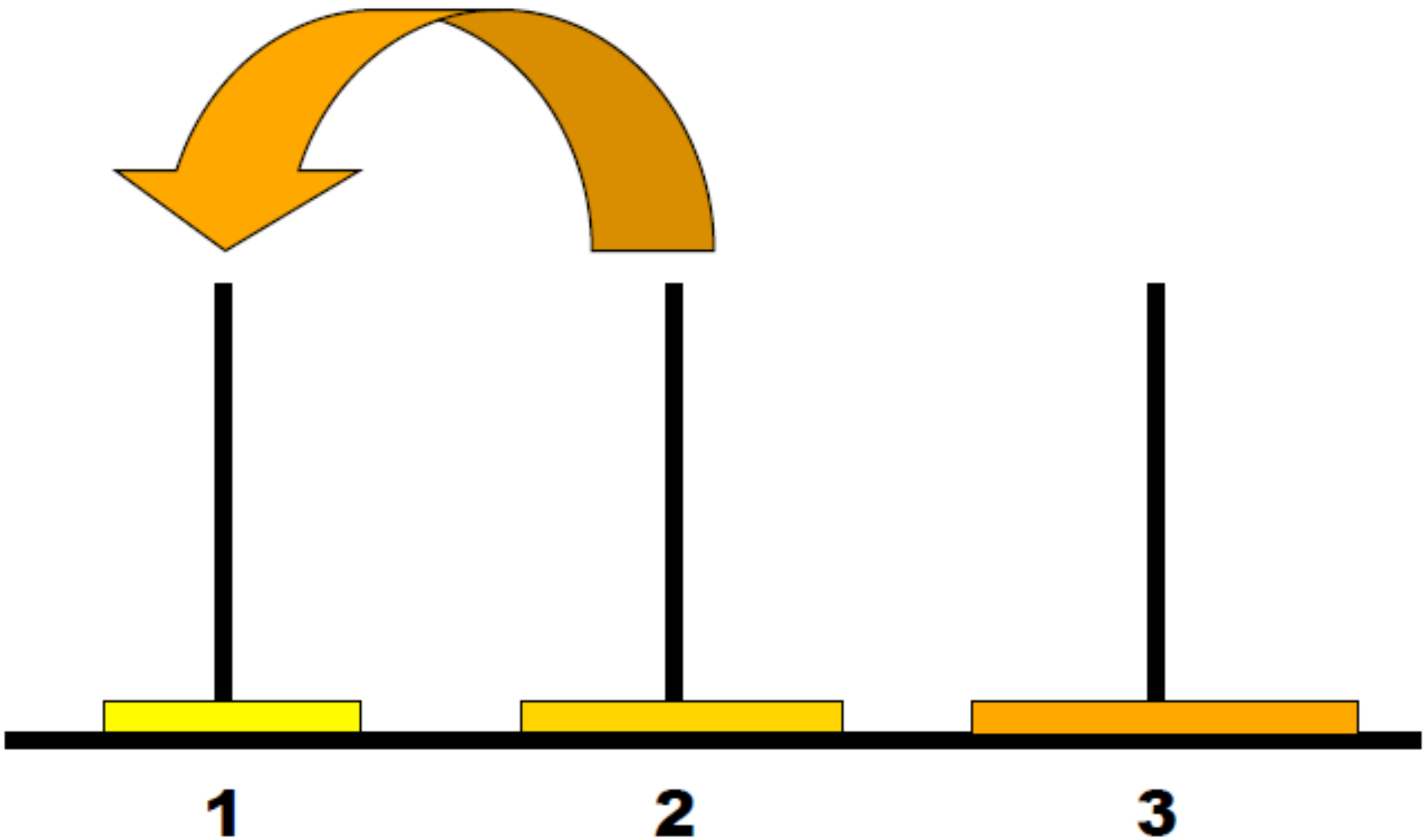
Move 3



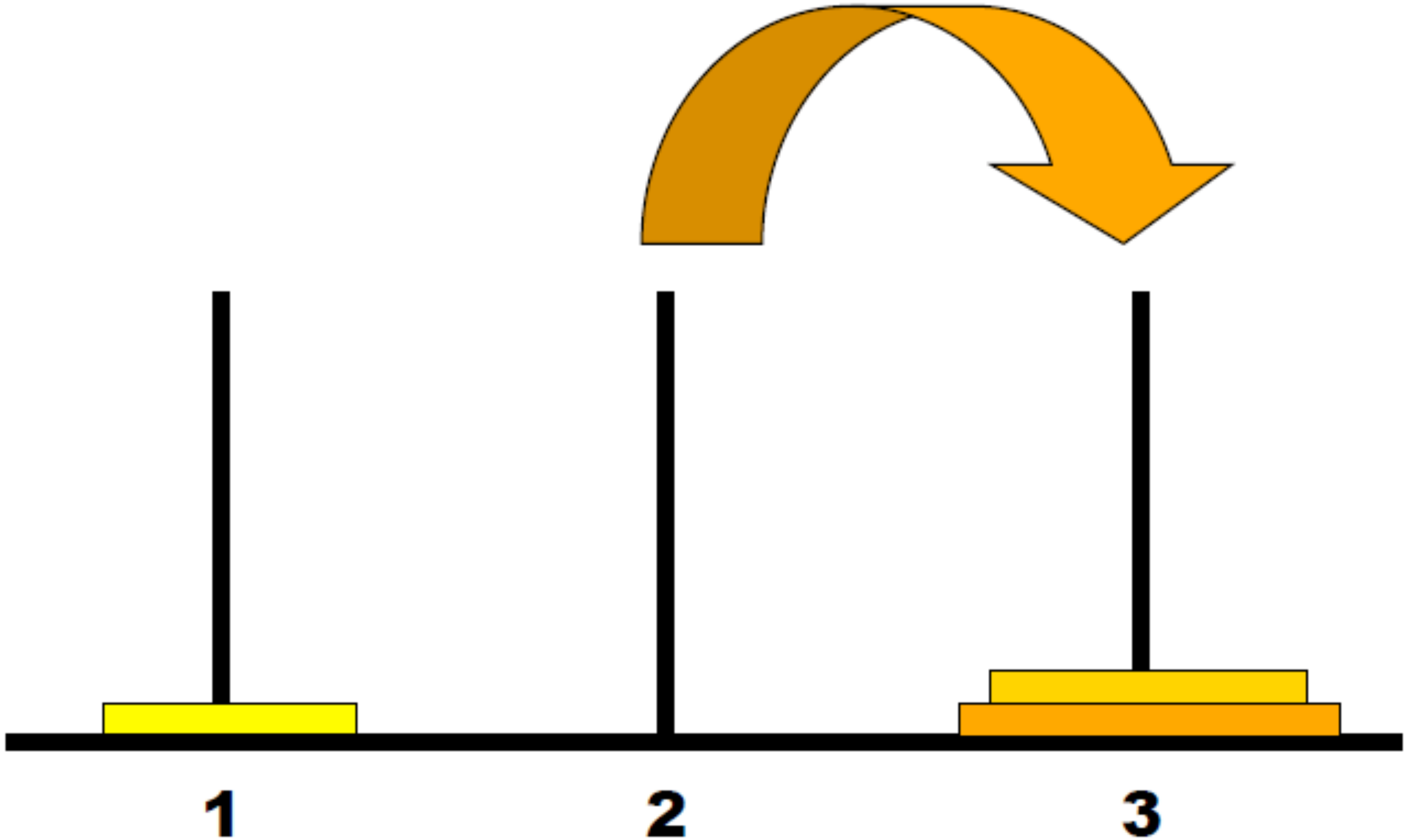
Move 4



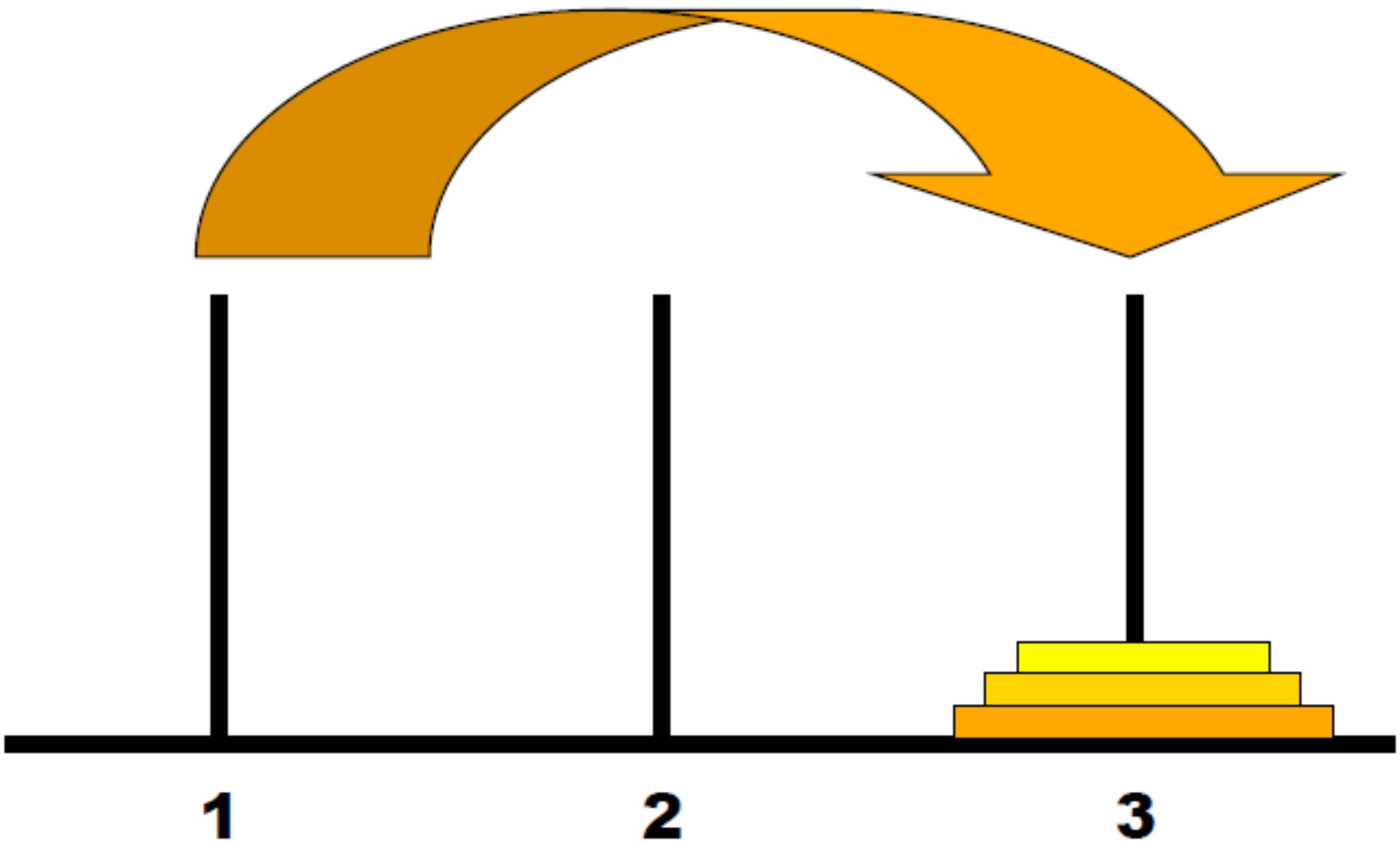
Move 5



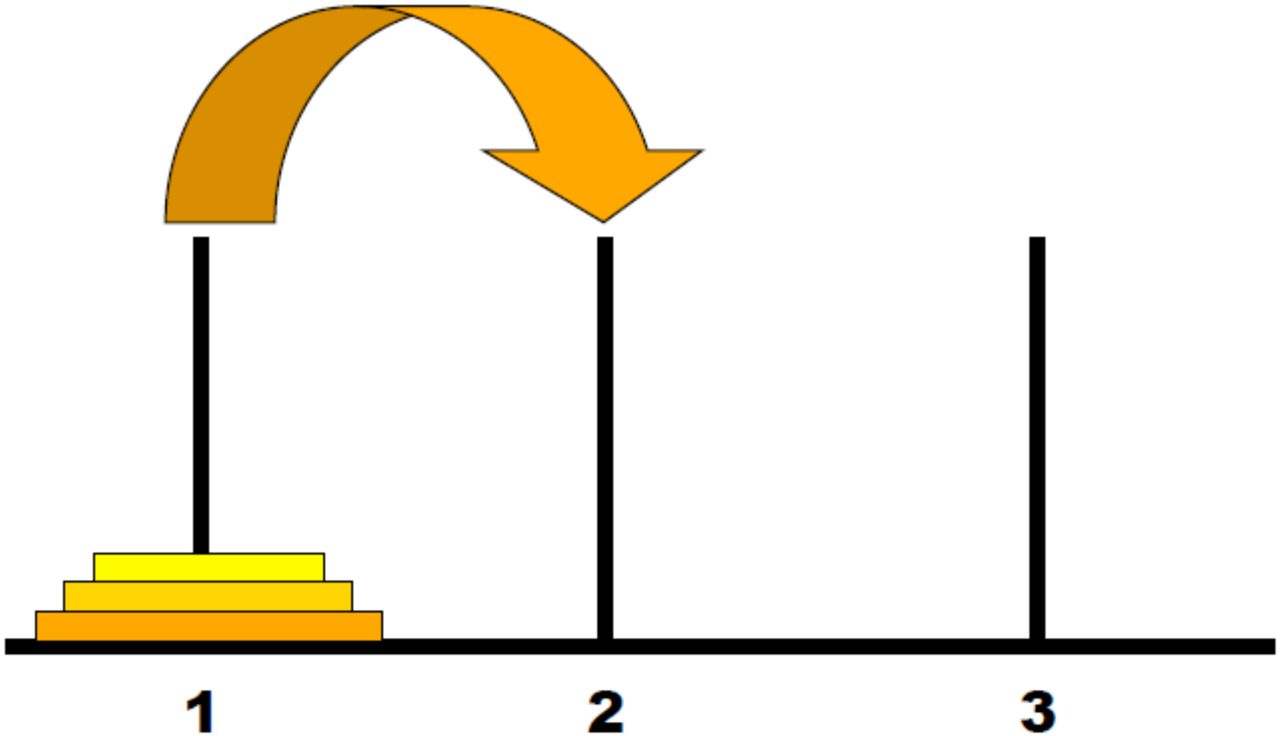
Move 6



Move 7

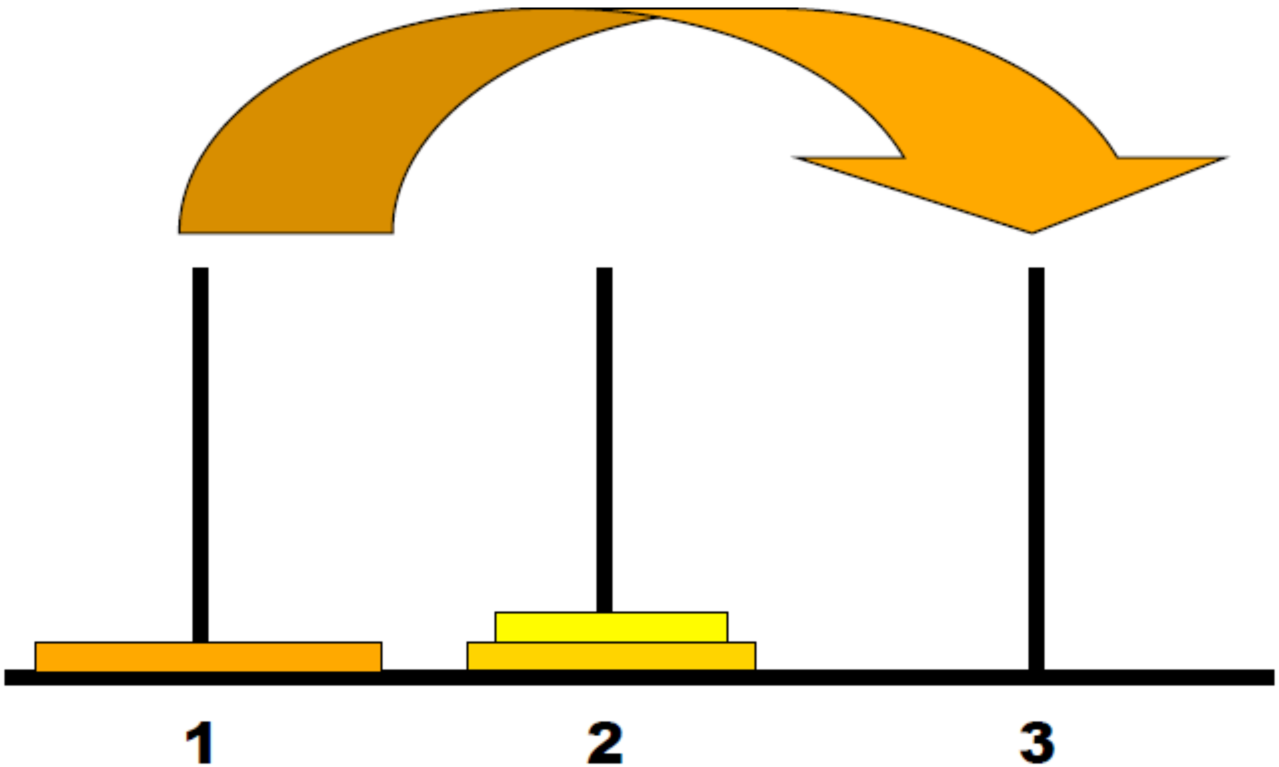


Simplifying the algorithm for 3 disks



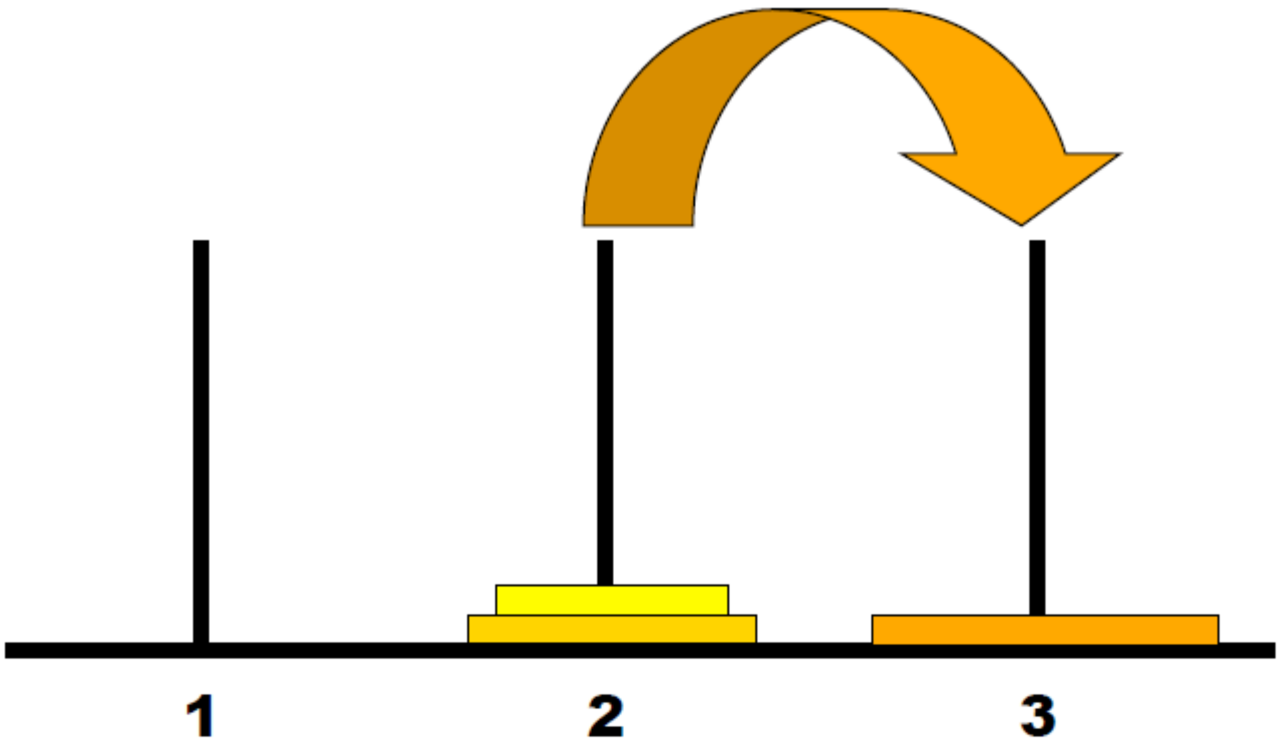
- Step 1. Move the top 2 disks from 1 to 2 using 3 as intermediate

Simplifying the algorithm for 3 disks



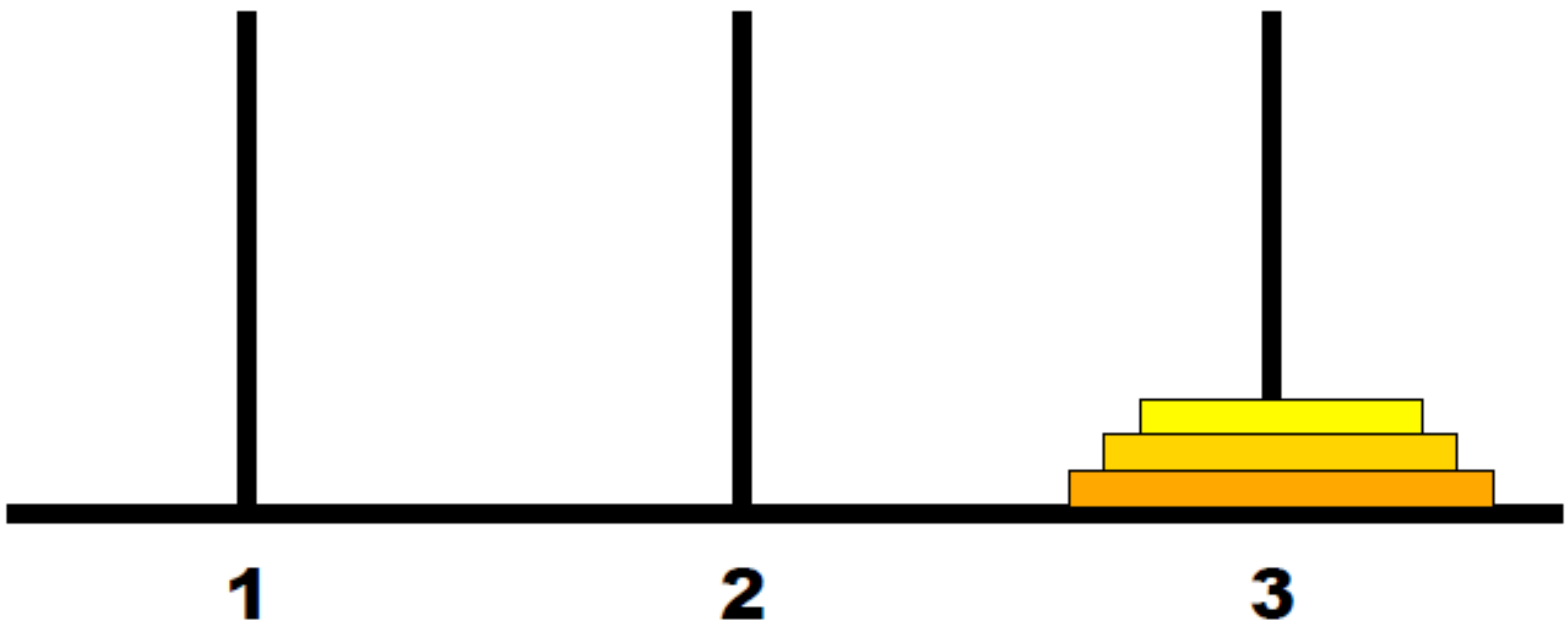
- Step 2. Move the remaining disk from 1 to 3

Simplifying the algorithm for 3 disks



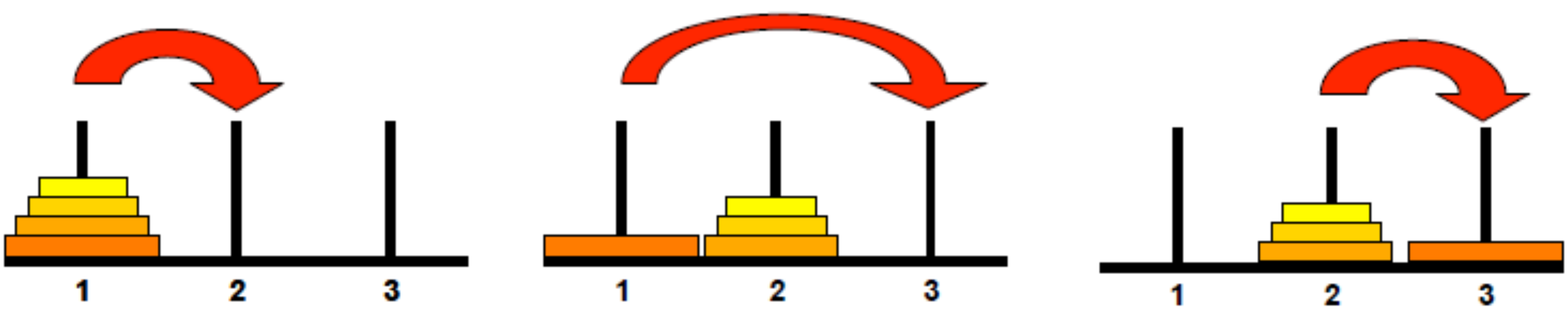
- Step 3. Move 2 disks from 2 to 3 using 1 as intermediate

Simplifying the algorithm for 3 disks



Recursive Towers of Hanoi

- At first glance, the recursive nature of the towers of Hanoi problem may not be obvious
- Consider, that the 3 disk problem must be solved as part of the 4 disk problem
- In fact it must be solved twice! Moving the bottom disk once in-between



The problem for 3 disks becomes

- A base case of a one-disk move from 1 to 3.
- A recursive step for moving 2 or more disks.
- To move *n disks from Peg 1 to Peg 3, we need to*
  - Move (*n-1*) *disks from Peg 1 to Peg 2*
    (Note: Peg 2 is the "unused" extra peg)
  - Move the *nth "bottom" disk from Peg 1 to Peg 3*
  - Move (*n-1*) *disks from Peg 2 to Peg 3*

Towers of Hanoi Algorithm

```
def towersOfHanoi(n, fromPeg, toPeg):
      if (n == 1):
          print "Move disk from peg",fromPeg,"to peg",toPeg
          return
unusedPeg = 6 - fromPeg - toPeg
towersOfHanoi(n-1,fromPeg,unusedPeg)
print "Move disk from peg", fromPeg,"to peg", toPeg
towersOfHanoi(n-1,unusedPeg,toPeg)
return
```

The number of disk moves is:

$$T(1) = 1$$

$$T(n) = 2T(n-1) + 1 = 2^n - 1$$

Exponential algorithm

Towers of Hanoi

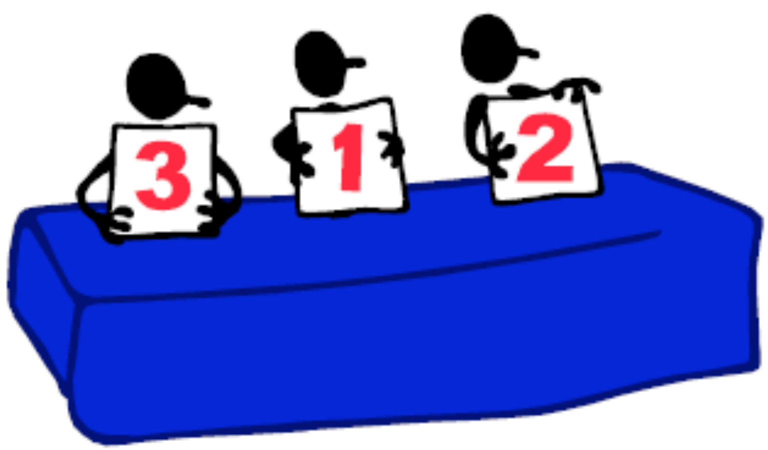
- If you call towerOfHanoi with ____ it takes ____
  - 1 disk ... 1 move
  - 2 disks ... 3 moves
  - 3 disks ... 7 moves
  - 4 disks ... 15 moves
  - 5 disks ... 31 moves
  - .
  - .
  - 20 disks ... 1,048,575 moves
  - 32 disks ... 4,294,967,295 moves

Another Algorithm: Sorting

- A very common problem is to arrange data into either ascending or descending order
  - **Viewing, printing**
  - **Faster to search, find min/max, compute median/mode, etc.**
- Lots of different sorting algorithms
  - **From the simple to very complex**
  - **Some optimized for certain situations (lots ofduplicates, almost sorted, etc.)**

Exercise

- You are given a list of 10 numbers {n1, n2, n3, n4, n5, n6, n7, n8, n9, n10}
- Write down precise detailed instructions for sorting them in ascending order



Sorting Exercise

- We'll look at your sorting algorithms more closely
- Are they correct?
- How many steps are used to sort N items?

How to Sort?

- How would you describe the task of sorting a list of numbers to a 5-year old, who knows only basic arithmetic operations?
- Goal 1: A correct algorithm
- There are many possible approaches
- Each requires the atomic operation of comparing two numbers
- Are all sorting approaches equal?
- What qualities distinguish “good” approaches from those less good?
  - Speed? Space required?

Selection Sort

Method #1

Find the smallest element and swap it with the first:

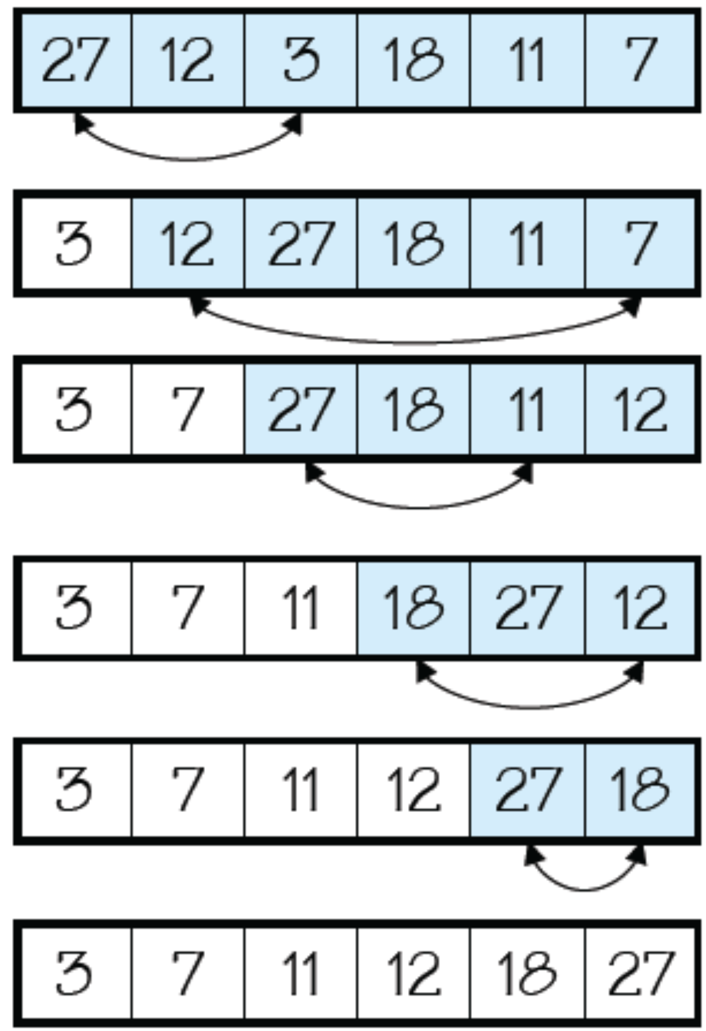
Find the next smallest element and swap it with the second:

Do the same for the third element:

And the fourth:

Finally, the fifth:

Completely sorted:



Selection Sort

```
def selectionSort(list):
    first = 0
    while (first < len(list)):
        index = findMin(list, first)
        temp = list[index]
        list[index] = list[first]
        list[first] = temp
        first = first+1
```

(n -1) swaps

$$\frac{n(n-1)}{2}$$ *comparisons*

```
def findMin(list,first):
    index = first
    for i in xrange(first+1,len(list)):
        if (list[i] < list[index]):
            index = I
    return index
```

Other Ways to Sort?

- Would you use this algorithm yourself?
  - Progress is slow, (i.e. moving one value to the front of the list after comparing to all others)
- Any Ideas?
- An Insertion Sort

Other Ways to Sort?

- Would you use this algorithm yourself?
  - Progress is slow, (i.e. moving one value to the front of the list after comapring to all others)
- Perhaps we can exploit recursion for sorting…
- Better yet, we can divide and conquer!

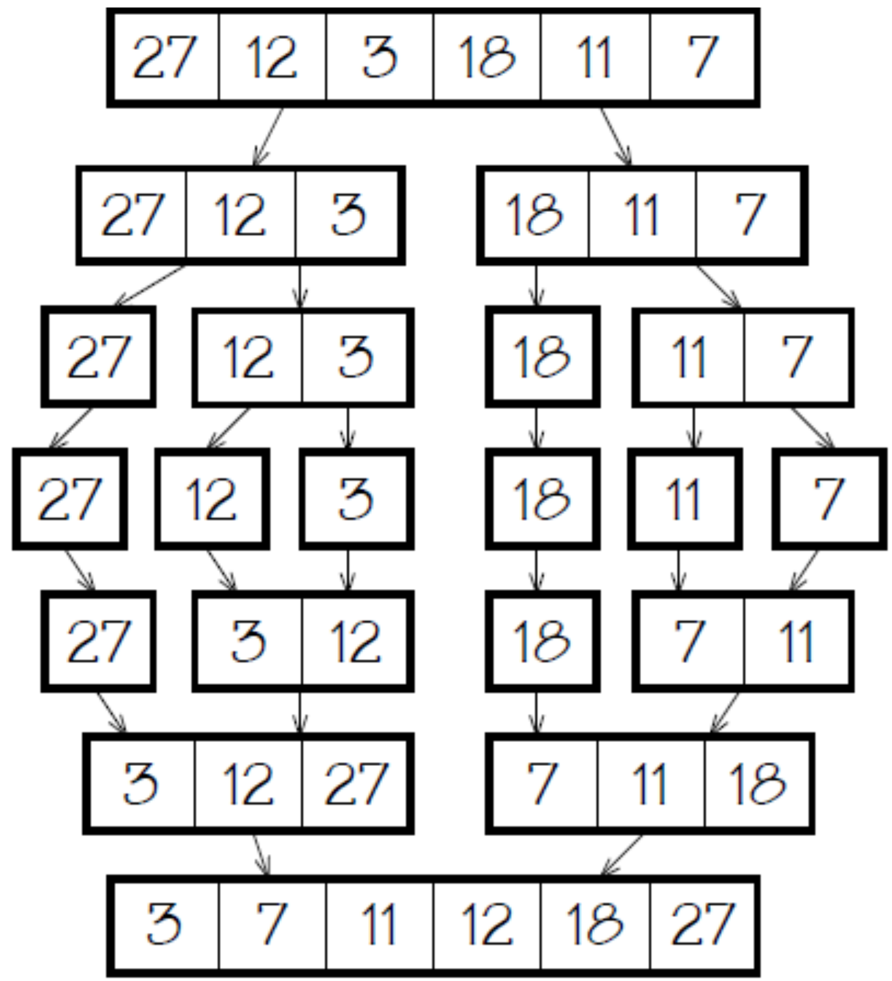
Merge Sort

Method #2

Split the list in half forming 2 sublists

Continue splitting sublists until lists are a just one item

Then combine sorted sublists together, by selecting the smallest value from the front of each sublist



Merge Sort

```
def mergeSort(list):
    if (len(list) == 1):
        return list
    half = len(list)/2
    left = mergeSort(list[:half])
    right = mergeSort(list[half:])
    return combine(left,right)
```

$\log_2(n)$ splits

< N steps to combine lists

```
def combine(listL,listR):
    mergedList = []
    while (len(listL) > 0 and len(listR) > 0):
        if (listL[0] < listR[0]):
            mergedList.append(listL.pop(0))
        else:
            mergedList.append(listR.pop(0))
    while (len(listL) > 0):
        mergedList.append(listL.pop(0))
    while (len(listR) > 0):
        mergedList.append(listR.pop(0))
    return mergedList
```

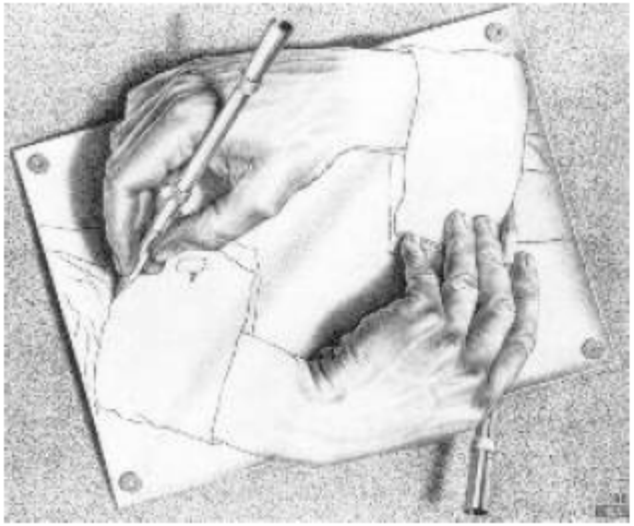
N(N-1)/2 vs N log2N

- For small numbers, perhaps not
  - N = 4, N(N-1)/2 = 6, N log2N = 8
  - N = 8, N(N-1)/2 = 28, N log2N = 24
  - N = 16, N(N-1)/2 = 120, N log2N = 64
- But the difference can be quite large for a large list of numbers
  - N = 1000, N(N-1)/2 = 499500, N log2N = 9966

Is Recursion the Secret Sauce?

- A noticeable difference between selection sort and merge sort, is that merge sort was specified

  as a recursive algorithm
- Does recursion always lead to fast algorithms?
- Previously, I offered recursion as a tool for specifying algorithms concisely, in terms of a common repeated "kernel"



Year 1202: Leonardo Fibonacci:

- He asked the following question:
  - How many pairs of rabbits are produced from a single pair in one year if every month each pair of rabbits more than 1 month old produces a new pair?
  - Here we assume that each pair has one male and one female, the rabbits never die, initially we have one pair which is less than 1 month old
  - *f(n): the number of pairs present at the beginning* of month *n*

Fibonacci Number

Fibonacci Number

- Clearly, we have:
  - *f(1) = 1 (the first pair we have)*
  - *f(2) = 1 (still the first pair we have because they are just 1* month old. They need to be more than one month old to reproduce)
  - *f(n) = f(n-1) + f(n-2) because f(n) is the sum of the old* rabbits from last month (*f(n-1)) and the new rabbits* reproduced from those *f(n-2) rabbits who are old enough to* reproduce.
  - *f: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …*
  - The solution for this recurrence is:

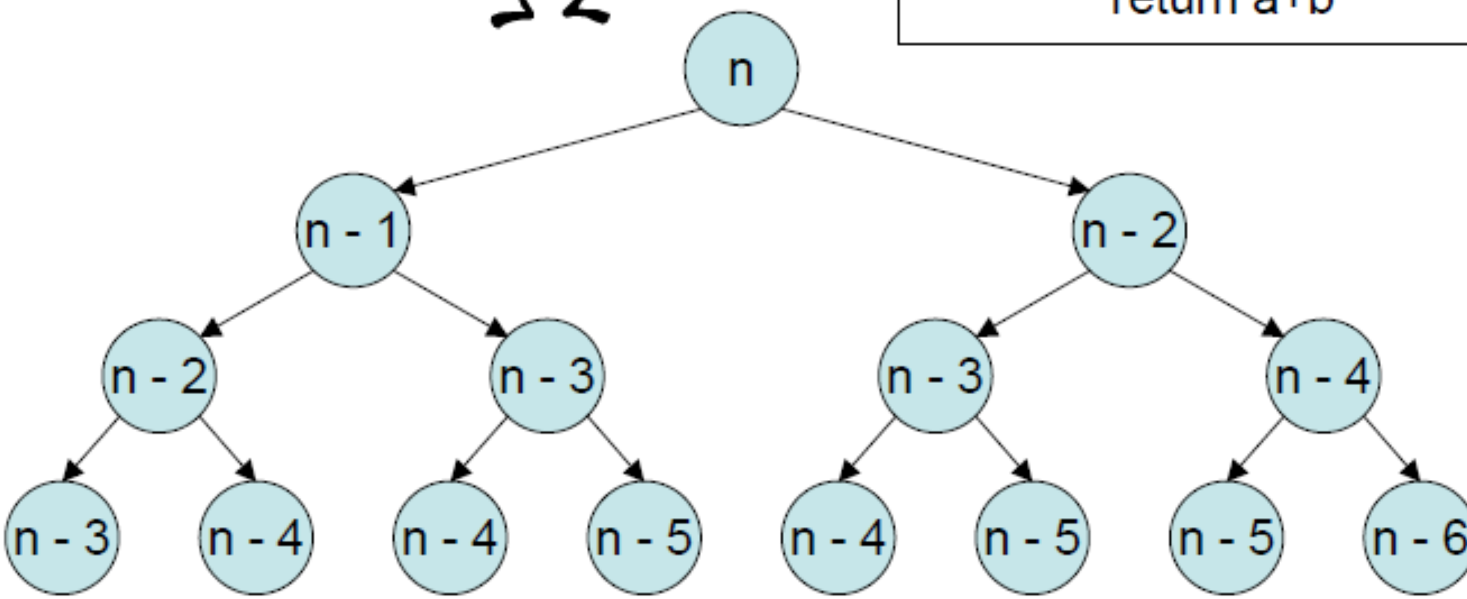
$$f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Fibonacci Number

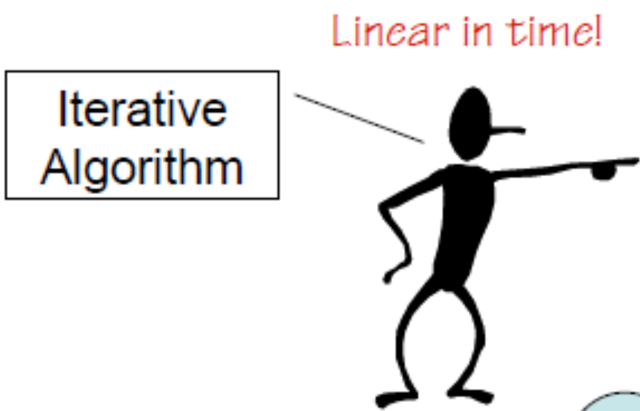
Exponential in time!

Recursive Algorithm

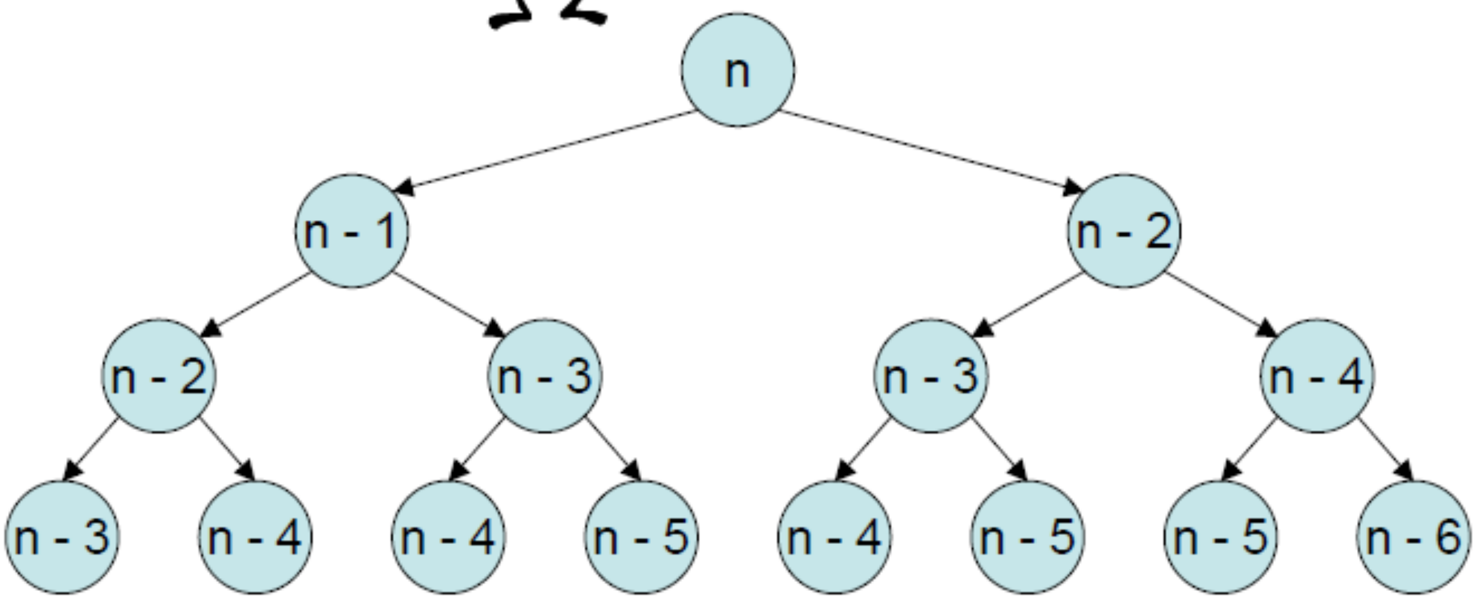
```
def fibonacciRecursive(n):
    if (n <= 2):
        return 1
    else:
        a = fibonacciRecursive(n-1)
        b = fibonacciRecursive(n-2)
        return a+b
```



Fibonacci Number




```
def fibonacci(n):
    f = [1,1]
    for i in xrange(2,n):
        f += [f[i-1]+f[i-2]]
    return f[n-1]
```



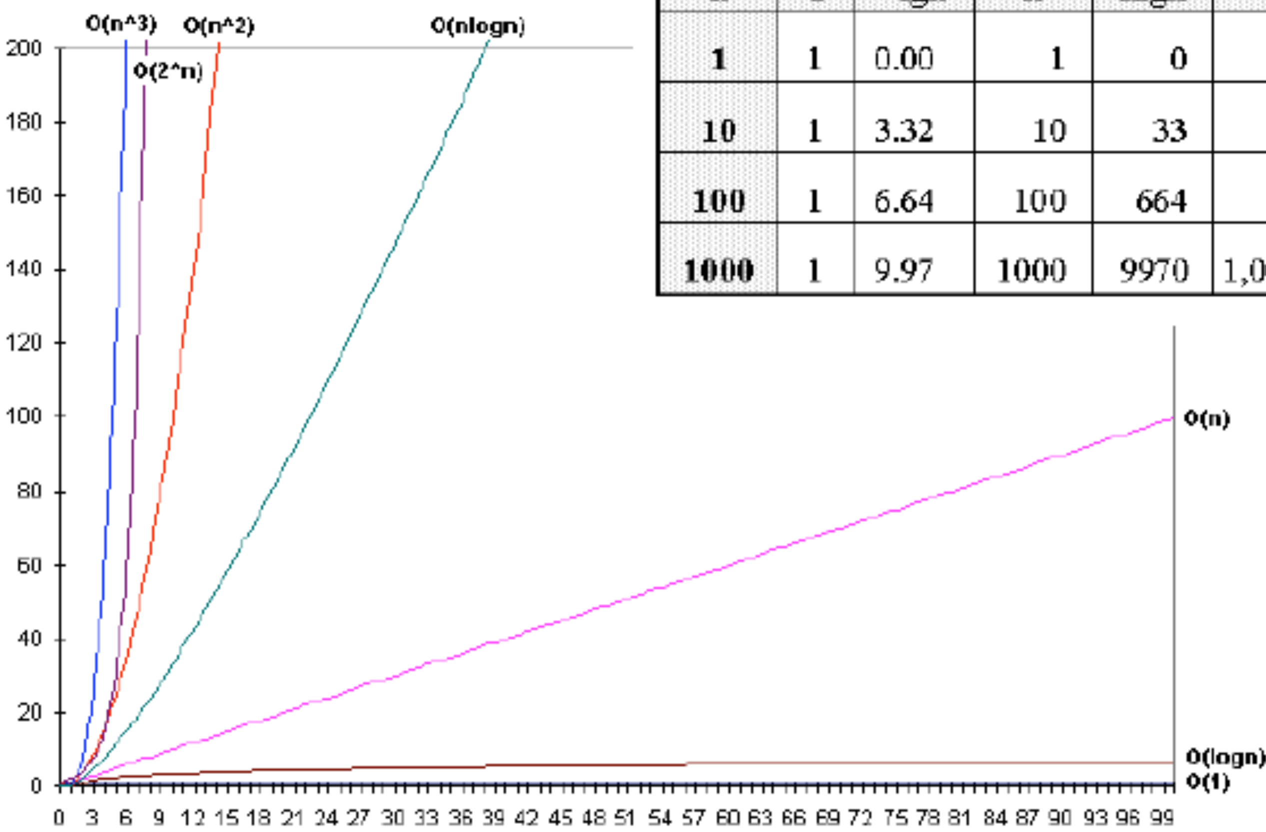
Is there a “Real difference”?

- 10’s Number of students in a class
- 100’s Number of students in a department
- 1000’s Number of students in the college of art and science
- 10000’s Number of students enrolled at UNC
- …
- …
- 10^10 Number of stars in the galaxy
- 10^20 Total number of all stars in the universe
- 10^80 Total number of particles in the universe
- 10^100 << Number of moves needed for 400 disks in the Towers of Hanoi puzzle

- Towers of Hanoi puzzle is *computable but it is NOT feasible.*

Is there a "Real" Difference?

- Growth of functions



n	1	lgn	n	nlgn	n^2	n^3	2^n
1	1	0.00	1	0	1	1	2
10	1	3.32	10	33	100	1,000	1024
100	1	6.64	100	664	10,000	1,000,000	1.2×10^{30}
1000	1	9.97	1000	9970	1,000,000	10^9	1.1×10^{301}

Asymptotic Notation

- *Order of growth is the interesting measure:*
  - Highest-order term is what counts
- As the input size grows larger it is the high order term that dominates
- Θ notation: $\Theta(n^2)$ = "*this function grows similarly to* n^2".
- Big-O notation: $O(n^2)$ = "*this function grows at* least as *slowly as* n^2".
  - Describes an *upper bound.*

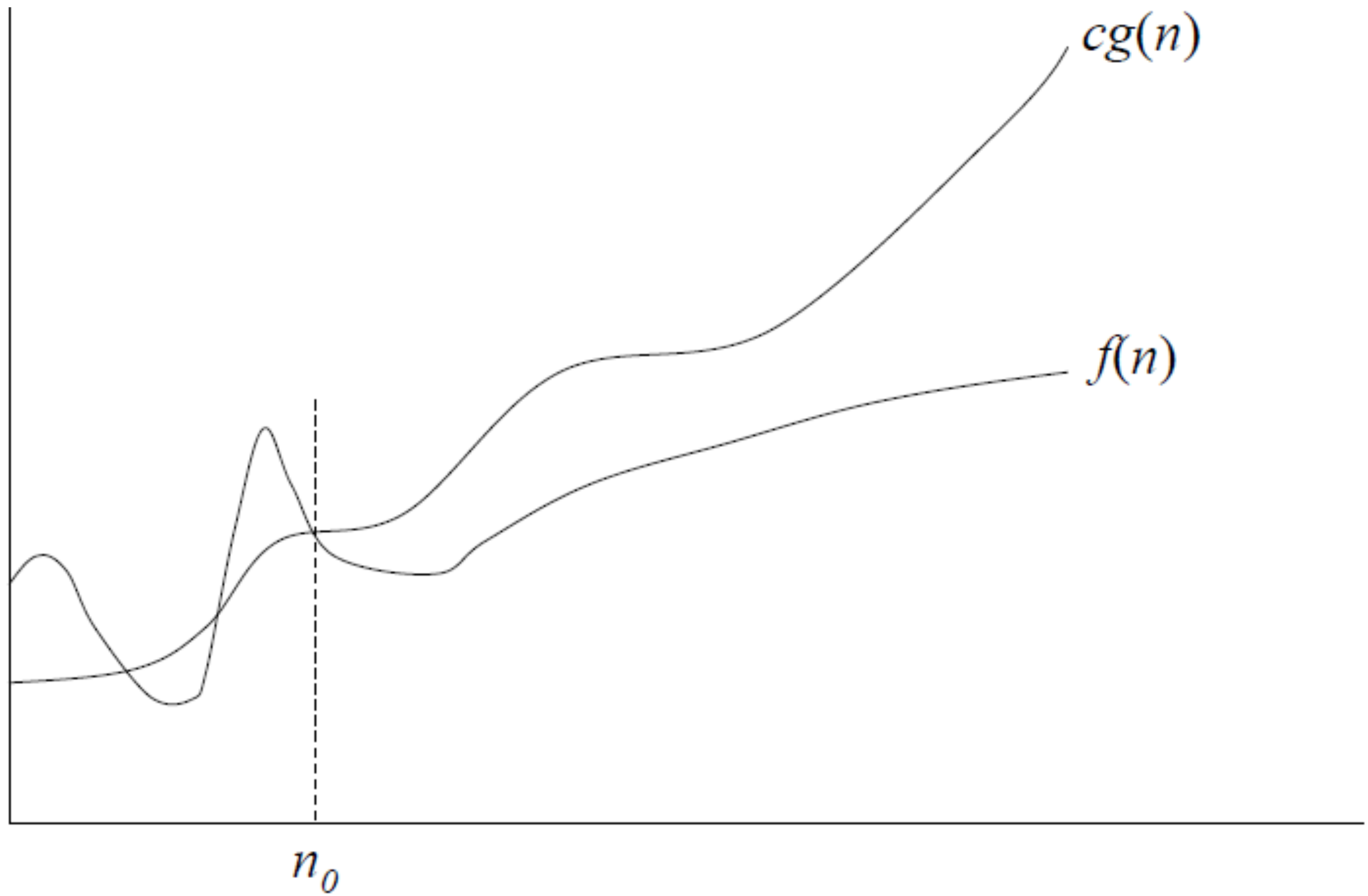
Big-O Notation

$f(n) = O(g(n))$: there exist positive constants c and n_0 such that

$$0 \le f(n) \le cg(n) \text{ for all } n \ge n_0$$

- What does it mean?
  - If $f(n) = O(n^2)$, *then:*
- *$f(n)$ can be larger than n^2 sometimes,* **but…**
- We can choose some constant *c and some value n0* such that for **every value of *n larger than n0* : *f (n)*** $< cn^2$
- That is, for values larger than n_0, *$f(n)$ is never more* than a constant multiplier greater than n^2
- Or, in other words, *$f(n)$ does not grow more than a* constant factor faster than n^2.

Visualization of *O(g(n))*



Big-O Notation

$$2n^2 = O(n^2)$$

$$1{,}000{,}000n^2 + 150{,}000 = O(n^2)$$

$$5n^2 - 7n + 20 = O(n^2)$$

$$2n^3 + 2 \neq O(n^2)$$

$$n^{2.1} \neq O(n^2)$$

Big-O Notation

- Prove that: $20n^2 + 2n + 5 = O(n^2)$
- Let $c = 21$ and $n_0 = 4$
- $21n^2 > 20n^2 + 2n + 5$ for all $n > 4$

  $n^2 > 2n + 5$ for all $n > 4$

  TRUE

Θ-Notation

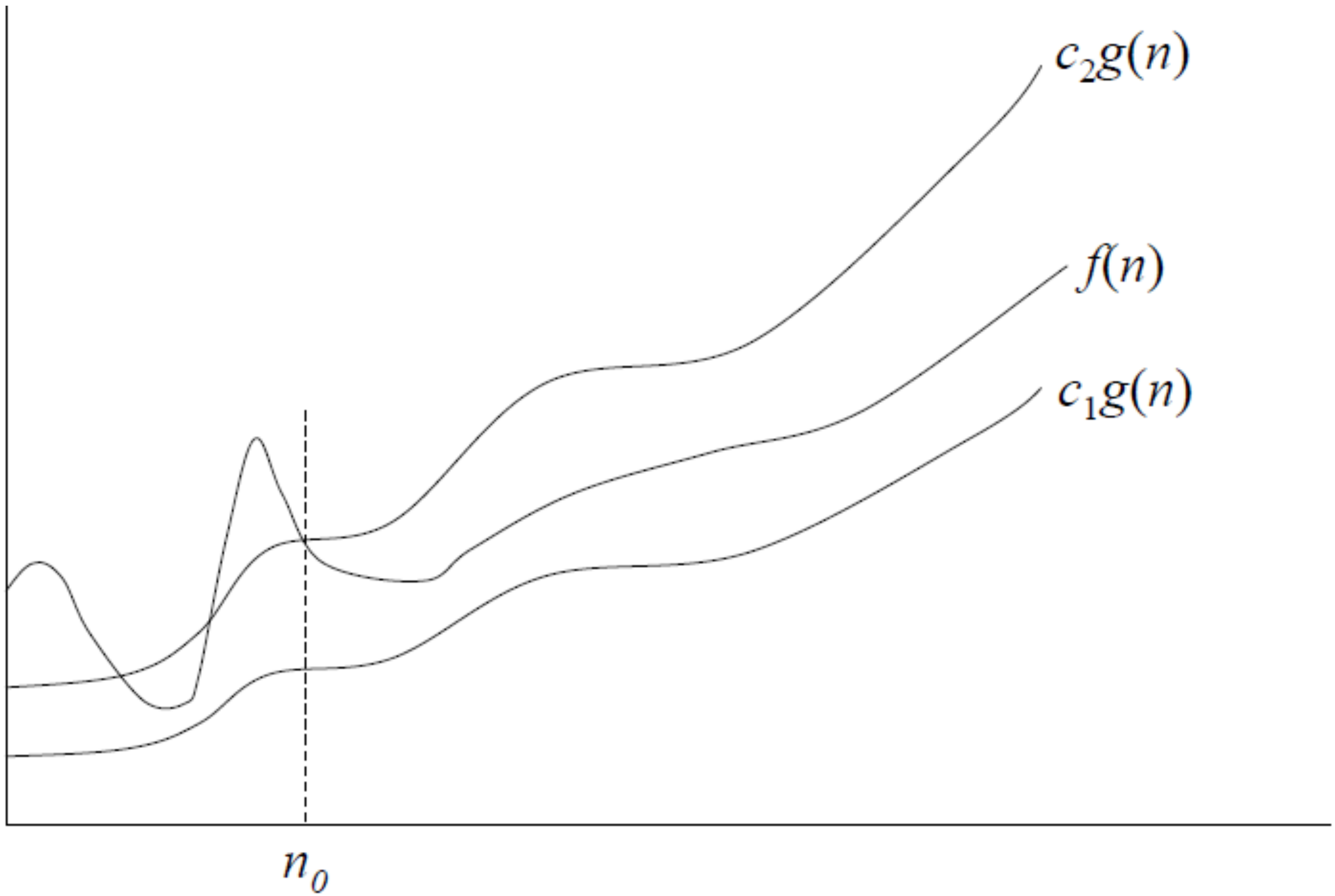
- Big-O is not a tight upper bound.
  In other words $n = O(n^2)$
- Θ provides a tight bound

$f(n) = \Theta(g(n))$: there exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0$$

- $n = O(n^2) \neq \Theta(n^2)$
- $200n^2 = O(n^2) = \Theta(n^2)$
- $n^{2.5} \neq O(n^2) \neq \Theta(n^2)$

Visualization of Θ(*g*(*n*))



Some Other Asymptotic Functions

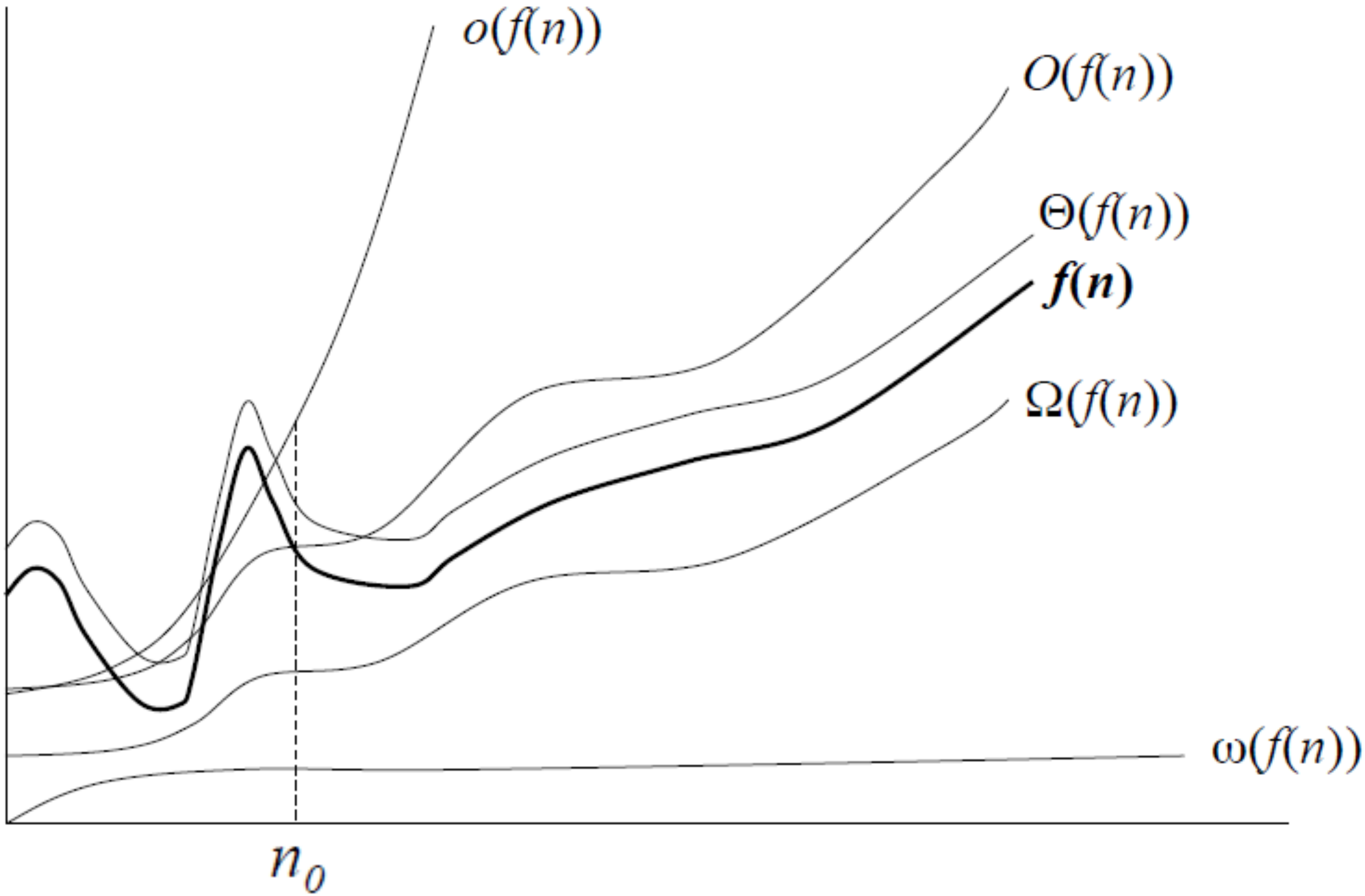
- Little o – A **non-tight** asymptotic upper bound
  - $n = o(n^2)$, $n = O(n^2)$
  - $3n^2 \neq o(n^2)$, $3n^2 = O(n^2)$
- Ω – A **lower** bound

$f(n) = \Omega(g(n))$: there exist positive constants c and n_0 such that

$$f(n) \geq cg(n) \text{ for all } n \geq n_0$$

  - $n^2 = \Omega(n)$
- ω – A **non-tight** asymptotic lower bound
- $f(n) = \Theta(n) \Leftrightarrow f(n) = O(n)$ **and** $f(n) = \Omega(n)$

Visualization of Asymptotic Growth



Analogy to Arithmetic Operators

$$f(n) = O(g(n)) \quad \approx \quad a \le b$$

$$f(n) = \Omega(g(n)) \quad \approx \quad a \ge b$$

$$f(n) = \Theta(g(n)) \quad \approx \quad a = b$$

$$f(n) = o(g(n)) \quad \approx \quad a < b$$

$$f(n) = \omega(g(n)) \quad \approx \quad a > b$$

Measures of Complexity

- Best case
  - **Super-fast in some limited situation is not very valuable information**
- Worst case
  - **Good upper-bound on behavior**
  - **Never get worse than this**
- Average case
  - **Averaged over all possible inputs**
  - **Most useful information about overall performance**
  - **Can be hard to compute precisely**

Complexity

- Time complexity is not necessarily the same as the space complexity
- Space Complexity: how much space an algorithm needs (as a function of *n*)
- Time vs. space

Techniques

- Algorithm design techniques
  - Exhaustive search
  - Greedy algorithms
  - Branch and bound algorithms
  - Dynamic programming
  - Divide and conquer algorithms
  - Randomized algorithms
- Tractable vs intractable algorithms