GENERIC RIGIDITY OF GRAPHS

1. Introduction and summary. In this survey, all graphs will be simple. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. A pair $F = (G, \mathbf{p})$ consisting of G together with an injective map $\mathbf{p} : V(G) \to \mathbb{R}^2$ is a **framework**, **linkage** or **skeletal structure** in the plane with **underlying graph** G. We visualize the vertices v_i of G as pivots at positions $\mathbf{p}(v_i)$, and the edges as rigid bars joining the corresponding pairs of pivots. In higher-dimensional frameworks we visualize the vertices as 'universal joints'.

We are going to define the concepts of mechanical rigidity and independence, and infinitesimal rigidity and independence. These concepts do not coincide, but there is a one-way implication: if a framework is infinitesimally rigid then it is mechanically rigid, and if it is infinitesimally independent then it is mechanically independent. These concepts do not depend just on the underlying graph: special geometry can cause a framework to be mechanically rigid when it "shouldn't" be, not mechanically rigid when it "should" be, or not infinitesimally rigid when it "should" be, although it cannot cause a framework to be infinitesimally rigid when it "shouldn't" be.

We call a framework **generic** when it has no special geometry. For generic frameworks, the mechanical and infinitesimal concepts coincide, and also depend only on the underlying graph, so that we can refer to the generic rigidity or generic independence of a *graph*. The aim is to give a combinatorial characterization of the graphs that are generically independent for dimension 2, which will automatically characterize the graphs that are generically rigid for dimension 2. We shall see eight different characterizations: two due (more or less) to L. Henneberg (1911) and J. E. Graver (1984), one due to G. Laman (1970), one due to L. Lovász and Y. Yemini (1982), and four due (directly or indirectly) to A. Dress (1987). It is known that none of these characterizations can extend in any obvious way to frameworks in four or more dimensions, and that five of them do not extend even to frameworks in three dimensions. But there are conjectures due to Henneberg–Graver and to Dress, each of which would provide a combinatorial characterization of generic independence in three dimensions. At the moment no such characterization is known, and finding one is the major unsolved problem of rigidity theory today.

2. Definitions. A **mechanical motion** of a framework $F = (G, \mathbf{p})$ is a parametrized family $(\mathbf{p}_t : 0 \leqslant t \leqslant 1)$ of maps such that:

(a) $\mathbf{p}_0 = \mathbf{p}$;
(b) the position $\mathbf{p}_t(v_i)$ of each vertex v_i is a differentiable function of t; and
(c) for each t and each edge $v_i v_j$ of G, the distance

$$\left\|\mathbf{p}_t(v_i) - \mathbf{p}_t(v_j)\right\| = \left\|\mathbf{p}(v_i) - \mathbf{p}(v_j)\right\|, \tag{1}$$

which is independent of t (so that the edges of F have 'constant length').

A **rigid motion** of F is a mechanical motion in which each framework $F_t = (G, \mathbf{p}_t)$ is geometrically congruent to F; that is, (1) holds for *every* i and j. If F does not admit any mechanical motions apart from its rigid motions, then F is **mechanically rigid**. If, for each edge e of F, $F - e$ admits a mechanical motion that is not admitted by F, then the edges of F are **mechanically independent**.

An **infinitesimal motion** of F is an assignment of a vector $\boldsymbol{\xi}_i$ to each vertex v_i such that the dot product

$$(\mathbf{p}(v_i) - \mathbf{p}(v_j)) \cdot (\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) = 0 \tag{2}$$

for each edge $v_i v_j$ of G (so that if each vertex v_i were simultaneously to start moving with velocity $\boldsymbol{\xi}_i$, then *momentarily* no bar would be stretched or compressed). By squaring and differentiating (1) when $t = 0$ we obtain

$$(\mathbf{p}(v_i)-\mathbf{p}(v_j))\cdot(\dot{\mathbf{p}}_0(v_i)-\dot{\mathbf{p}}_0(v_j)) = 0;$$

that is, the initial velocities of the vertices in a mechanical motion of F constitute an infinitesimal motion of F. The framework F_3 in Fig. 1 (taken from Graver (1984)) shows that the converse is false: the assignment of a non-zero vector to vertex b, perpendicular to edge bc, and $\mathbf{0}$ to every other vertex, gives an infinitesimal motion that does not correspond to any mechanical motion. However, the converse is true for *generic* frameworks: see below.

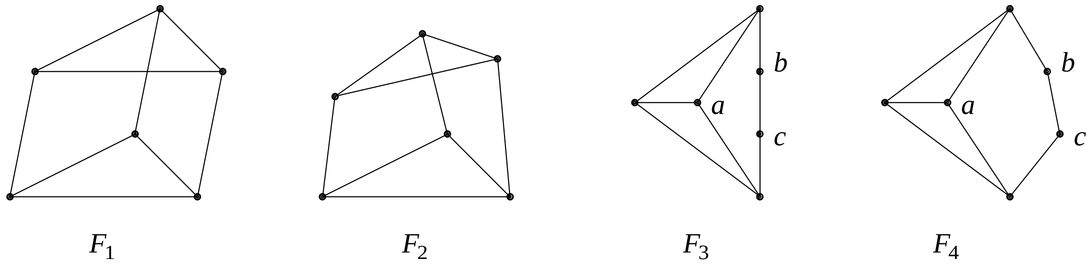

Fig. 1.

If F has n vertices and m edges, then the equations (2) can be written in coordinate form as a system of m linear equations in $2n$ real variables (the 2 coordinates of each of the n vectors $\boldsymbol{\xi}_i$). We can write this system in the form

$$M\boldsymbol{\xi} = \mathbf{0}, \tag{3}$$

where each edge of F contributes one row to the $m \times 2n$ matrix M. For example, if F consists of K_4 with its vertices at positions $(x_1, y_1), \ldots, (x_4, y_4)$ and 'velocity' vectors $(\xi_1, \eta_1), \ldots, (\xi_4, \eta_4)$, then (3) becomes

$$\begin{bmatrix} x_1-x_2 & y_1-y_2 & x_2-x_1 & y_2-y_1 & 0 & 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & 0 & 0 & x_3-x_1 & y_3-y_1 & 0 & 0 \\ x_1-x_4 & y_1-y_4 & 0 & 0 & 0 & 0 & x_4-x_1 & y_4-y_1 \\ 0 & 0 & x_2-x_3 & y_2-y_3 & x_3-x_2 & y_3-y_2 & 0 & 0 \\ 0 & 0 & x_2-x_4 & y_2-y_4 & 0 & 0 & x_4-x_2 & y_4-y_2 \\ 0 & 0 & 0 & 0 & x_3-x_4 & y_3-y_4 & x_4-x_3 & y_4-y_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \\ \xi_3 \\ \eta_3 \\ \xi_4 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The infinitesimal motions of F are precisely the solutions of (3). A set of edges of F is **infinitesimally independent** if the corresponding rows of M are linearly independent. The initial velocities of the *rigid* motions of F satisfy (3) and form a vector space of dimension 3. F is **infinitesimally rigid** if these are the *only* solutions of (3); that is, if $\operatorname{rank} M = 2n-3$.

These concepts depend on the map $\mathbf{p}$ as well as on the underlying graph G. Because of its special geometry, the framework F_1 in Fig. 1 is not rigid, either mechanically or infinitesimally, whereas F_2, with the same underlying graph, is both mechanically and infinitesimally rigid. Again, because of its special geometry, F_3 is mechanically rigid but not infinitesimally rigid, for the reason described earlier; whereas F_4, with the same underlying graph as F_3, is not even mechanically rigid. However, $F_4 \cup \{ac\}$ is both infinitesimally and mechanically rigid, whereas $F_3 \cup \{ac\}$ is still not infinitesimally rigid.

The (set of vertices of a) framework $F = (G, \mathbf{p})$ with n vertices is **generic** if, in the matrix M corresponding to the complete framework $(K_n, \mathbf{p})$ with the same vertices as F, every submatrix has the largest possible rank that it can have. This is equivalent to saying that the determinant of a square submatrix of M is never zero except when it is *identically* zero when regarded as a polynomial in the variables x_i and y_i. Note that almost all frameworks are generic, in the sense that there are generic frameworks arbitrarily close to any given framework, and if F is generic, then there exists an ε such that if F' is obtained from F by displacing the vertices in arbitrary directions by less than ε, then F' is also generic.

It is obvious from the definition that any two *generic* frameworks with the same underlying graph are either both infinitesimally rigid or both infinitesimally non-rigid, and that their edge-sets are either both infinitesimally independent or both infinitesimally dependent; in other words, for *generic* frameworks, infinitesimal independence and infinitesimal rigidity depend only on the underlying graph G. Also, L. Asimow and B. Roth (1978, 1979) proved that if F is generic then every infinitesimal motion of F consists of the initial velocities of some mechanical motion, so that for generic frameworks the mechanical and infinitesimal concepts coincide. We say that a graph G is **generically rigid** for dimension 2 if every 2-dimensional generic framework with underlying graph G is infinitesimally (or mechanically) rigid, and G (or a set of edges in G) is **generically independent** for dimension 2 if, for every 2-dimensional generic framework F with underlying graph G, F (or the appropriate set of edges in F) is infinitesimally independent. It is clear that a graph with n vertices is generically rigid for dimension 2 if and only if it contains a set of $2n-3$ edges that are generically independent for dimension 2.

3. Matrix methods. The following lemmas were apparently proved by L. Henneberg (1911) and rediscovered independently by J. E. Graver (1984).

Lemma 1. If x is a vertex of a graph G with degree $d(x) \leqslant 2$, and $G-x$ is generically independent for dimension 2, then so is G.

Proof. It suffices to prove the result when $d(x) = 2$. So let the vertices of G be $v_1, \ldots, v_n$, where $x = v_n$ and $N(x) = \{v_1, v_2\}$. Let $F = (G, \mathbf{p})$ be a generic framework with underlying graph G, and suppose that $\mathbf{p}(v_i) = (x_i, y_i)$ for each i. The matrix M representing F in equation (3) can be written in the form

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_m \end{array} \left[\begin{array}{cccccccc|cc} x_1-x_n & y_1-y_n & 0 & 0 & 0 & \ldots & 0 & & x_n-x_1 & y_n-y_1 \\ 0 & 0 & x_2-x_n & y_2-y_n & 0 & \ldots & 0 & & x_n-x_2 & y_n-y_2 \\ \hline & & & & & & & & 0 & 0 \\ & & & M' & & & & & \vdots & \vdots \\ & & & & & & & & 0 & 0 \end{array} \right]$$

where M' is the matrix representing $G' = G-x$. Since $G-x$ is generically independent, the rows of M' are linearly independent. We want to prove that the rows of M are linearly independent. Suppose that some linear combination $\sum a_i R_i$ of the rows of M is equal to the zero vector. Since F is generic,

$$\begin{vmatrix} x_n-x_1 & y_n-y_1 \\ x_n-x_2 & y_n-y_2 \end{vmatrix} \neq 0,$$

and so $a_1 = a_2 = 0$. We are left with a linear combination of the rows of M', which are linearly independent; so $a_3 = \ldots = a_m = 0$. Thus the rows of M are linearly independent, as required. □

If G is generically independent and $X \subseteq V(G)$, the **relative degree of freedom** of X in G is the largest number of edges $\{e_1, \ldots, e_r\}$ that one can add between pairs of non-adjacent vertices of X so that $G \cup \{e_1, \ldots, e_r\}$ is still generically independent.

Lemma 2. If x is a vertex of a graph G with degree 3, $G-x$ is generically independent for dimension 2, and $N(x)$ has at least one relative degree of freedom in $G-x$, then G is generically independent for dimension 2.

Proof. Let the vertices of G be $v_1,\ldots,v_n$, where $x = v_n$ and $N(x) = \{v_1, v_2, v_3\}$, and suppose w.l.o.g. that $v_1v_2 \notin G$ and $(G-x)\cup\{v_1v_2\}$ is generically independent. Let $F = (G, \mathbf{p})$ be a generic framework with underlying graph G, and suppose that $\mathbf{p}(v_i) = (x_i, y_i)$ for each i. The matrix M representing F in equation (3) can be written in the form

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_m \end{array}\left[\begin{array}{cccccccccc|cc} x_1-x_n & y_1-y_n & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & & x_n-x_1 & y_n-y_1 \\ 0 & 0 & x_2-x_n & y_2-y_n & 0 & 0 & 0 & \ldots & 0 & & x_n-x_2 & y_n-y_2 \\ 0 & 0 & 0 & 0 & x_3-x_n & y_3-y_n & 0 & \ldots & 0 & & x_n-x_3 & y_n-y_3 \\ \hline & & & & & & & & & & 0 & 0 \\ & & & & M' & & & & & & \vdots & \vdots \\ & & & & & & & & & & 0 & 0 \end{array}\right]$$

where M' is the matrix representing $G' = G-x$. Suppose that some linear combination $\sum a_i R_i$ of the rows of M is equal to the zero vector. Then $a_1R_1 + a_2R_2 + a_3R_3 \in V'$, the space spanned by the rows of M' (with two extra zeros added on the end). Since F is generic, a_1, a_2, a_3 are uniquely determined up to a constant multiple, and we may suppose that

$$a_1 = \begin{vmatrix} x_n-x_2 & y_n-y_2 \\ x_n-x_3 & y_n-y_3 \end{vmatrix}, \qquad a_2 = -\begin{vmatrix} x_n-x_1 & y_n-y_1 \\ x_n-x_3 & y_n-y_3 \end{vmatrix}, \qquad a_3 = \begin{vmatrix} x_n-x_1 & y_n-y_1 \\ x_n-x_2 & y_n-y_2 \end{vmatrix}.$$

Let $\mathbf{v}(\mathbf{p}) := a_1R_1 + a_2R_2 + a_3R_3$. Keeping $\mathbf{p}(v_1),\ldots,\mathbf{p}(v_{n-1})$ fixed and allowing $\mathbf{p}(v_n)$ to vary, we see that $\mathbf{v}(\mathbf{p}) \in V'$ whenever $\mathbf{p}(v_1),\ldots,\mathbf{p}(v_n)$ are generic. However, this is true for points $\mathbf{p}(v_n)$ arbitrarily close to any given point (x_0, y_0), and so, since V' is a closed set (in the analytic sense), it follows that $\mathbf{v}(\mathbf{p}) \in V'$ for *every* choice of $\mathbf{p}(v_n)$. Taking $\mathbf{p}(v_n) = \frac{1}{2}(\mathbf{p}(v_1) + \mathbf{p}(v_2))$, we find that $a_3 = 0$ and

$$a_1 = a_2 = \frac{1}{2}\begin{vmatrix} x_1-x_2 & y_1-y_2 \\ x_1-x_3 & y_1-y_3 \end{vmatrix} \neq 0$$

since $\mathbf{p}(v_1),\ldots,\mathbf{p}(v_{n-1})$ are generic, and so $\mathbf{p}(v_n) = \frac{1}{2}a_1\mathbf{e}_{12}$, where

$$\mathbf{e}_{12} = (x_1-x_2,\ y_1-y_2,\ x_2-x_1,\ y_2-y_1,\ 0,\ \ldots,\ 0),$$

which is the row vector corresponding to the edge v_1v_2. Thus $(G-x)\cup\{v_1v_2\}$ is generically dependent, contrary to hypothesis. This contradiction shows that the rows of M are linearly independent, as required. □

It is easy to see from the above proofs that Lemmas 1 and 2 extend to an arbitrary dimension q, with $d(x) \leqslant q$ in Lemma 1 and $d(x) = q+1$ in Lemma 2. In dimension 3, this covers the cases $d(x) \leqslant 4$. The case $d(x) = 5$ is the subject of the following conjecture, due to Henneberg and Graver. However, as H. Maehara (to appear) pointed out, this conjecture does not extend in the obvious way to dimension 4: $K_{6,6}$ is a counterexample.

Conjecture. If x is a vertex of a graph G with degree 5, $G-x$ is generically independent for dimension 3, $N(x)$ has at least two relative degrees of freedom in $G-x$, and every subset of 4 vertices of $N(x)$ has at least one relative degree of freedom in $G-x$, then G is generically independent for dimension 3.

4. Characterizations. Let $\mathcal{G}$ be a class of graphs (meaning that if a graph belongs to $\mathcal{G}$ then so do all graphs isomorphic to it). $\mathcal{G}$ is **hereditary** if every proper subgraph of a graph in $\mathcal{G}$ is in $\mathcal{G}$. If $\mathcal{G}$ is hereditary, we shall call the graphs in $\mathcal{G}$ **$\mathcal{G}$-independent** and all other graphs **$\mathcal{G}$-dependent**. By a **minimal** $\mathcal{G}$-dependent graph, we mean a graph that is not in $\mathcal{G}$, all of whose proper subgraphs are in $\mathcal{G}$. A hereditary class $\mathcal{G}$ is **matroidal** if, whenever G_1 and G_2 are distinct minimal $\mathcal{G}$-dependent graphs that both contain the same edge e, then $(G_1 \cup G_2) - e \notin \mathcal{G}$.

Let $\mathcal{G}_q$ denote the class of graphs that are generically independent for dimension q. If M is the matrix in (3) corresponding to the graph $G_1 \cup G_2$, then G_i is a minimal $\mathcal{G}_q$-dependent graph if and only if the rows of M corresponding to the edges of G_i form a minimal linearly dependent set of rows; from this it is easy to see that the class $\mathcal{G}_q$ is matroidal.

The following two characterizations are essentially due to J. E. Graver (1984).

Theorem 1. $\mathcal{G}_2$ is the unique non-empty hereditary class $\mathcal{G}$ satisfying:
A1. Every graph in $\mathcal{G}$ contains a vertex with degree at most 3.
A2. If $x \in V(G)$, $d(x) \leqslant 2$ and $G - x \in \mathcal{G}$, then $G \in \mathcal{G}$.
A3. If $x \in V(G)$ and $d(x) = 3$, then $G \in \mathcal{G}$ if and only if $(G - x) \cup \{e\} \in \mathcal{G}$ for some edge e joining two non-adjacent vertices of $N(x)$.

Theorem 2. $\mathcal{G}_2$ is the unique non-empty matroidal class $\mathcal{G}$ not containing K_4 and satisfying A1, A2 and the ‘if’ part of A3.

The proof of these results uses:

Lemma 3. Let $\mathcal{G}$ be a matroidal class of graphs, let $H \in \mathcal{G}$, and let G_1 be a minimal $\mathcal{G}$-dependent graph that contains exactly one edge e' that is not in H. Then $(H - e) \cup \{e'\} \in \mathcal{G}$, for each edge e in $G_1 - e'$.

Proof. Suppose not. Let G_2 be a minimal $\mathcal{G}$-dependent subgraph of $(H - e) \cup \{e'\}$. If $e' \notin G_2$ we get a contradiction because then $G_2 \subset H \in \mathcal{G}$, and if $e' \in G_2$ then the matroidal property of $\mathcal{G}$ ensures that $(G_1 \cup G_2) - e' \notin \mathcal{G}$, which is a contradiction since $(G_1 \cup G_2) - e' \subseteq H \in \mathcal{G}$. □

The next result is the well known characterization by G. Laman (1970).

Theorem 3. A graph G belongs to $\mathcal{G}_2$ if and only if, for each subgraph H of G with at least two vertices,

$$|E(H)| \leqslant 2|V(H)| - 3. \qquad (\mathrm{L}_2)$$

Note that (L_2) holds for each such subgraph of G if and only if it holds for each such *induced* subgraph.

From this we obtain the following characterization due to L. Lovász and Y. Yemini (1982).

Theorem 4. A graph G belongs to $\mathcal{G}_2$ if and only if, for each edge e of G, doubling e results in a multigraph G' that is the union of two forests.

Proof. C. St J. A. Nash-Williams (1964) proved that a multigraph G' is the union of k forests if and only if, for each submultigraph H' of G', $|E(H')| \leqslant k(|V(H')| - 1)$. Thus G' is the union of two forests if and only if, for each submultigraph H' of G',

$$|E(H')| \leqslant 2|V(H')| - 2. \qquad (4)$$

But if (L_2) holds for every subgraph H of G, then clearly (4) holds for every submultigraph H' of G' (taking H to be the corresponding subgraph of G, so that $V(H) = V(H')$ and $|E(H)| \geqslant |E(H')| - 1$). And if (L_2) fails for some subgraph H of G, then choosing $e \in H$ and taking H' to be the corresponding subgraph of G', so that $V(H') = V(H)$ and $|E(H')| = |E(H)| + 1$, we see that (4) fails. So the result follows from Theorem 3. □

For our final set of four characterizations, we need several definitions and lemmas. The **$\mathcal{G}$-rank** $r_{\mathcal{G}}(G)$ of a graph G is the largest number of edges in any $\mathcal{G}$-independent subgraph of G.

Lemma 4. Let $\mathcal{G}$ be a matroidal class of graphs, G an arbitrary graph, and e_1, e_2 edges not in G such that $r_{\mathcal{G}}(G \cup \{e_1\}) = r_{\mathcal{G}}(G \cup \{e_2\}) = r_{\mathcal{G}}(G)$. Then $r_{\mathcal{G}}(G \cup \{e_1, e_2\}) = r_{\mathcal{G}}(G)$.

We shall say that a graph G is **$\mathcal{G}$-rigid** if $r_{\mathcal{G}}(G \cup \{e\}) = r_{\mathcal{G}}(G)$ whenever e is a new edge joining two non-adjacent vertices of G, and **$\mathcal{G}$-closed** if $r_{\mathcal{G}}(G \cup \{e\}) > r_{\mathcal{G}}(G)$ whenever e is a new edge joining two non-adjacent vertices of $G \cup \bar{K}_2$. Clearly all complete graphs are $\mathcal{G}$-rigid, and a $\mathcal{G}$-rigid subgraph of a $\mathcal{G}$-closed graph G induces a complete subgraph of G. Also, a graph is $\mathcal{G}_2$-independent or $\mathcal{G}_2$-rigid if and only if it is generically independent or generically rigid for dimension 2.

We shall say that a matroidal class $\mathcal{G}$ of graphs is **matroidal for dimension q** if it satisfies the two conditions:

B1. If G_1 and G_2 are two $\mathcal{G}$-independent graphs with at most $q-1$ vertices in common, then $G_1 \cup G_2 \cup \{uw\}$ is $\mathcal{G}$-independent whenever $u \in V(G_1) \setminus V(G_2)$ and $w \in V(G_2) \setminus V(G_1)$.

B2. If G_1 and G_2 are two $\mathcal{G}$-rigid graphs with at least q vertices in common, then $G_1 \cup G_2$ is $\mathcal{G}$-rigid.

Lemma 5. The class $\mathcal{G}_q$ is matroidal for dimension q.

Lemma 6. Let $\mathcal{G}$ be a non-empty class of graphs that is matroidal for dimension 2.

(a) $r_{\mathcal{G}}(K_n) = 2n-3$ if $n \geqslant 2$.

(b) If $G \in \mathcal{G}$ then $|E(H)| \leqslant 2|V(H)| - 3$ for every subgraph H of G with at least two vertices.

(c) If G is $\mathcal{G}$-closed and G_1 and G_2 are distinct maximal cliques of G, then $|V(G_1) \cap V(G_2)| \leqslant 1$.

Now consider the following four conditions.

C1. If $G \in \mathcal{G}$ and u, w are non-adjacent vertices of G such that $G \cup \{uw\} \notin \mathcal{G}$, then G has a $\mathcal{G}$-rigid subgraph H such that $u, w \in V(H)$.

C2. If G is a minimal $\mathcal{G}$-dependent graph, then $G - e$ is $\mathcal{G}$-rigid, for each edge e of G.

C3. If G is a minimal $\mathcal{G}$-dependent graph, then G is $\mathcal{G}$-rigid.

C4. If G is $\mathcal{G}$-closed and $F_1, \ldots, F_k$ are the maximal cliques of G, then $r_{\mathcal{G}}(G) = r_{\mathcal{G}}(F_1) + \ldots + r_{\mathcal{G}}(F_k)$.

Lemma 7. Let $\mathcal{G}$ be a class of graphs that is matroidal for dimension 2. Then C1 $\Leftrightarrow$ C2 $\Rightarrow$ C3 $\Rightarrow$ C4.

We are at last in a position to prove the final four characterizations. The one using C4 was obtained by J. E. Graver (to appear) following an observation by A. Dress (1987).

Theorem 5. $\mathcal{G}_2$ is the unique non-empty class $\mathcal{G}$ of graphs that is matroidal for dimension 2 and satisfies any one of the conditions C1, C2, C3 and C4 (in which case it satisfies them all).

Corollary 5.1. If a 2-dimensional generic framework determines the distance apart of two vertices u and w, then it has a rigid subframework containing u and w.

Proof. This is effectively C1 for the class $\mathcal{G}_2$. □

5. One-dimensional and three-dimensional frameworks. It is easy to see that every one-dimensional framework (with distinct vertices) is generic, and that a one-dimensional framework is rigid if and only if it is connected and is independent if and only if it is circuit-free. Thus we have the following theorems, which are directly analogous to Theorems 1–5.

Theorem 1_1. $\mathcal{G}_1$ is the unique non-empty hereditary class $\mathcal{G}$ satisfying:

A1_1. Every graph in $\mathcal{G}$ contains a vertex with degree at most 1.

A2_1. If $x \in V(G)$, $d(x) \leqslant 1$ and $G - x \in \mathcal{G}$, then $G \in \mathcal{G}$.

Theorem 2_1. $\mathcal{G}_1$ is the unique non-empty matroidal class $\mathcal{G}$ satisfying A1_1 and A2_1.

Theorem 3_1. A graph G belongs to $\mathscr{G}_1$ if and only if, for each subgraph H of G,

$$|E(H)| \leqslant |V(H)| - 1. \qquad (\mathrm{L}_1)$$

Note that (L_1) holds for each such subgraph of G if and only if it holds for each such *induced* subgraph.

Theorem 4_1. A graph G belongs to $\mathscr{G}_1$ if and only if it is a forest.

Theorem 5_1. $\mathscr{G}_1$ is the unique non-empty class $\mathscr{G}$ of graphs that is matroidal for dimension 1 and satisfies any one of the conditions C1, C2, C3 and C4 (in which case it satisfies them all).

Little has been proved about three-dimensional frameworks. Theorems 3 and 3_1 suggest the following:

False Analogue of Laman's Theorem (FALT). A graph G belongs to $\mathscr{G}_3$ if and only if, for each subgraph H of G with more than one edge,

$$|E(H)| \leqslant 3|V(H)| - 6. \qquad (\mathrm{L}_3)$$

The graph G_3 in Fig. 2 shows that this is false. Every subgraph H of G_3 with more than one edge satisfies (L_3), but $G_3 \notin \mathscr{G}_3$ because it has exactly $3|V(G_3)| - 6$ edges and so cannot be $\mathscr{G}_3$-independent without being $\mathscr{G}_3$-rigid, which it clearly isn't. Laman's theorem can be interpreted as saying that there are exactly two reasons why a 2-dimensional framework $F = (G, \mathbf{p})$ can be infinitesimally or mechanically dependent: either some subgraph H of G has too many edges for its number of vertices (specifically, (L_2) fails), or there is some special geometry that causes F to be dependent (that is, F is not generic—Laman's theorem does not attempt to classify the different types of special geometry that can arise here). FALT would imply that only the same two reasons could apply in three dimensions, whereas the graph G_3 shows that there is another reason, namely, that G consists of two graphs H_1 and H_2 joined at two vertices u and w that are not adjacent in either H_1 or H_2 but whose distance apart in F is determined by both H_1 and H_2. This reason can evidently apply in any dimension, but it is covered by conditions (L_1) and (L_2) in dimensions 1 and 2. G_4 in Fig. 2 is the analogue of G_3 in two dimensions, but it is $\mathscr{G}_2$-rigid and violates Laman's condition (L_2), whereas G_3 is evidently not $\mathscr{G}_3$-rigid and it satisfies (L_3). Probably there are no other reasons why a 3-dimensional framework can be dependent; but nobody has yet found a precise formulation of this intuitive statement.

A second reason why FALT fails is that the class $\mathscr{G}$ of graphs that it describes is not even matroidal. For example, if G_1 and G_2 are the graphs in Fig. 2, having precisely one edge e in common, then G_1 and G_2 are minimal $\mathscr{G}$-dependent graphs, but $(G_1 \cup G_2) - e = G_3 \in \mathscr{G}$. Perhaps $\mathscr{G}_3$ is the largest matroidal subclass of $\mathscr{G}$; but, even if it is, nobody seems to know how to characterize it.

A third way of seeing that FALT is false is to observe that it would imply that $\mathscr{G}_3$ satisfies condition C1, so that the analogue of Corollary 5.1 holds in three dimensions. However, the

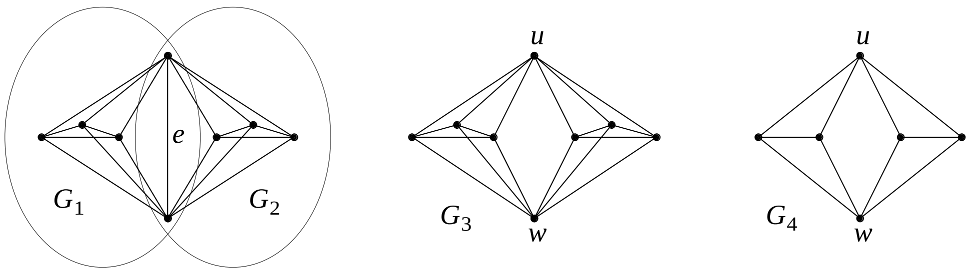

Fig. 2.

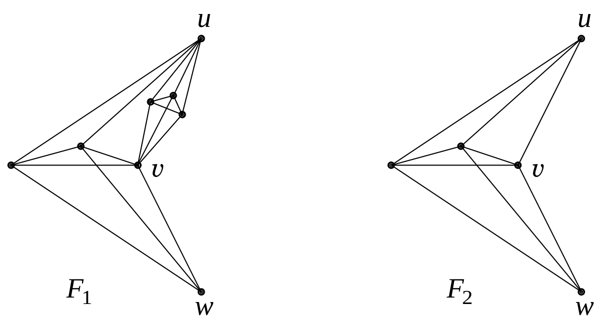

Fig. 3.

3-dimensional framework F_1 in Fig. 3 is a counterexample. It is formed from the rigid framework F_2 by replacing the edge uv by a 'spindle' isomorphic to F_2. So far as the rest of the framework is concerned, this spindle has the same effect as the edge uv, namely, to determine the distance apart of u and v; thus F_1 determines the distance apart of u and w, just as F_2 does. But, unlike the edge, the spindle can rotate on its axis, and so F_1 is not rigid, and it has no rigid subframework containing u and w. The framework obtained from F_1 by adding the edge uw is a minimal dependent framework that is not rigid, which shows that $\mathcal{G}_3$ does not satisfy condition C2 or C3 either.

Thus Theorem 3 and the conditions C1, C2 and C3 in Theorem 5 do not extend to three dimensions in any obvious way, and it seems unlikely that the closely related Theorem 4 can extend either. However, the Henneberg−Graver conjecture in Section 3 would give rise to theorems that are natural generalizations of Theorems 1 and 2 to three dimensions. And A. Dress (1987) has made the following conjecture, which is a refinement (hopefully, the appropriate refinement) of condition C4.

Conjecture. If G is $\mathcal{G}_3$-closed and $F_1,\ldots,F_k$ are the maximal cliques of G, then $r_{\mathcal{G}}(G) = \sum_{i=1}^{k} r_{\mathcal{G}}(F_i) - \sum_{e\in E(G)} \rho_e$ where ρ_e is one less than the number of maximal cliques containing e.

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GENERIC RIGIDITY OF GRAPHS

1. Introduction and summary.

2. Definitions.

3. Matrix methods.

4. Characterizations.

Proof of Theorems 1 and 2. We have seen that $\mathcal{G}_2$ is a matroidal class. Also, a graph in $\mathcal{G}_2$ with n vertices has at most $2n-3$ edges, whence it is not K_4 and contains a vertex with degree $\leqslant 3$. Since A2 and the 'if' part of A3 follow from Lemmas 1 and 2, we have proved all of Theorem 2 except for the uniqueness of the class $\mathcal{G}$ described therein. On the other hand, there is clearly at most one non-empty hereditary class $\mathcal{G}$ satisfying A1–A3, and so if we can prove that any non-empty matroidal class $\mathcal{G}$ not containing K_4 and satisfying A1, A2 and the 'if' part of A3 also satisfies the 'only if' part of A3, then we will have proved both of Theorems 1 and 2.

So let $\mathcal{G}$ be such a class. Suppose that $G \in \mathcal{G}$, $x \in V(G)$, $d(x) = 3$ and $(G-x) \cup \{e\} \notin \mathcal{G}$ for each edge e joining two non-adjacent vertices of $N(x)$. Suppose that, among all such counterexamples, G has as many edges as possible joining pairs of vertices in $N(x)$. Since $K_4 \notin \mathcal{G}$ and $G \in \mathcal{G}$, $K_4 \not\subseteq G$, and so some two vertices of $N(x)$ are non-adjacent: let e be a new edge joining them. Since $(G-x) \cup \{e\} \notin \mathcal{G}$, there is a minimal $\mathcal{G}$-dependent graph G_1 such that $G_1 \subseteq (G-x) \cup \{e\}$. Since $G \in \mathcal{G}$, $e \in G_1$. Let e' be an edge of G_1 that does not join two vertices in $N(x)$, which exists since $K_3 \in \mathcal{G}$ by A2. By Lemma 3, $G' := (G-e') \cup \{e\} \in \mathcal{G}$. Since G' has more edges than G joining pairs of vertices in $N(x)$, G' is not a counterexample to 'only if' in A3, and so we can add an edge f joining two vertices of $N(x)$ that are non-adjacent in G' so that $H := (G'-x) \cup \{f\} \in \mathcal{G}$. Then G_1 contains exactly one edge e' not in H, and $(H-e) \cup \{e'\} = (G-x) \cup \{f\} \notin \mathcal{G}$, contradicting Lemma 3. $\square$

Proof of Theorem 3. Let $\mathcal{G}$ be the class of graphs characterized in the theorem. It is easy to see that $\mathcal{G}$ satisfies A1, A2 and the 'if' part of A3. (In A3, (L_2) will hold for any subgraph H of G obtained from $H-x$ by adding x and at most two edges. And if H contains three edges not in $H-x$ then $|E(H-x)| \leqslant 2|V(H-x)|-4$ because $(H-x) \cup \{e\}$ satisfies (L_2).) So, by Theorem 2, it suffices to prove that $\mathcal{G}$ is matroidal.

To do this, let G_1 and G_2 be distinct minimal graphs not in $\mathcal{G}$, so that

$$|E(G_1)| = 2|V(G_1)|-2$$

and

$$|E(G_2)| = 2|V(G_2)|-2.$$

Since $G_1 \cap G_2 \subsetneqq G_1$, $G_1 \cap G_2 \in \mathcal{G}$ and so

$$|E(G_1 \cap G_2)| \leqslant 2|V(G_1 \cap G_2)|-3,$$

so that

$$|E(G_1 \cup G_2)| \geqslant 2|V(G_1 \cup G_2)|-2-2+3$$

and

$$|E((G_1 \cup G_2)-e)| \geqslant 2|V((G_1 \cup G_2)-e)|-2.$$

Thus $(G_1 \cup G_2) - e \notin \mathcal{G}$. □

Proof of Lemma 4. Suppose $r_{\mathcal{G}}(G \cup \{e_1, e_2\}) > r_{\mathcal{G}}(G)$. Let H and H' be largest $\mathcal{G}$-independent subgraphs of $G \cup \{e_1, e_2\}$ and G, chosen to have as many edges as possible in common, where evidently $e_1, e_2 \in H$ and $|E(H)| = |E(H')| + 1$. Then $\exists\ e'$ in $H' \setminus H$. Since $H \cup \{e'\} \notin \mathcal{G}$, $H \cup \{e'\}$ contains a minimal $\mathcal{G}$-dependent graph G_1. Clearly $e' \in G_1$. Choose e in $G_1 \setminus H'$, which exists since $G_1 \not\subseteq H' \in \mathcal{G}$. By Lemma 3, $H_1 := (H - e) \cup \{e'\} \in \mathcal{G}$. But H_1 has more edges in common with H' than H has, $\Rightarrow\Leftarrow$. □

Proof of Lemma 6. (a) Since $\mathcal{G} \neq \emptyset$, $K_1 \in \mathcal{G}$, and B1 easily implies that $K_2 \in \mathcal{G}$. By taking $G_2 := K_2$ in B1 we see that B1 $\Rightarrow$ A2. $K_2 + \bar{K}_{n-2}$ is a spanning subgraph of K_n, where '+' denotes 'join'. It is easy to see that $K_2 + \bar{K}_{n-2}$ is $\mathcal{G}$-independent, by A2, and $\mathcal{G}$-rigid, by B2 (since K_3 is $\mathcal{G}$-rigid). By repeated application of Lemma 4, $r_{\mathcal{G}}(K_n) = |E(K_2 + \bar{K}_{n-2})| = 2n - 3$.

(b) If $G \in \mathcal{G}$ and $H \subseteq G$ $(H \neq K_1)$, then H is a $\mathcal{G}$-independent subgraph of $K_{|V(H)|}$ and so $|E(H)| \leqslant 2|V(H)| - 3$ by (a).

(c) G_1 and G_2 are cliques, hence $\mathcal{G}$-rigid, and so if $|V(G_1) \cap V(G_2)| \geqslant 2$ then $G_1 \cup G_2$ is $\mathcal{G}$-rigid by B2. But $G_1 \cup G_2$ is not complete, and so G is not $\mathcal{G}$-closed, $\Rightarrow\Leftarrow$. □

Proof of Lemma 7. C1 $\Rightarrow$ C2: Applying C1 to $G - e$ we see that $G - e$ has a $\mathcal{G}$-rigid subgraph H that contains both end vertices of e. Then $H \cup \{e\}$ is $\mathcal{G}$-dependent. Since G is a minimal $\mathcal{G}$-dependent graph, $H \cup \{e\} = G$ and $G - e = H$ is $\mathcal{G}$-rigid.

C2 $\Rightarrow$ C1: Let H' be a minimal $\mathcal{G}$-dependent subgraph of $G \cup \{uw\}$, necessarily containing the edge uw. Then $H = H' - uw$ is $\mathcal{G}$-rigid by C2.

C2 $\Rightarrow$ C3: Obvious.

C3 $\Rightarrow$ C4: By Lemma 6(c), $E(F_i) \cap E(F_j) = \emptyset$ whenever $i \neq j$. Let B be a largest $\mathcal{G}$-independent subgraph of G. Then

$$r_{\mathcal{G}}(G) = |E(B)| = \sum_i |E(B \cap F_i)| \leqslant \sum_i r_{\mathcal{G}}(F_i)$$

since $B \cap F_i \subseteq B \in \mathcal{G}$. Now let B_i be a largest $\mathcal{G}$-independent subgraph of F_i, for each i, and $B := B_1 \cup \ldots \cup B_k$. Suppose B is $\mathcal{G}$-dependent and let H be a minimal $\mathcal{G}$-dependent subgraph of B. By C3, H is $\mathcal{G}$-rigid, whence $H \subseteq F_i$ for some i. But then $H \subseteq B_i \in \mathcal{G}$, which is impossible. Thus $B \in \mathcal{G}$ and

$$r_{\mathcal{G}}(G) \geqslant |E(B)| = \sum_i |E(B_i)| = \sum_i r_{\mathcal{G}}(F_i),$$

whence C4 holds. □

Proof of Theorem 5. $\mathcal{G}_2$ is matroidal for dimension 2 by Lemma 5. To see that it satisfies C2, let G be a minimal $\mathcal{G}_2$-dependent graph with n vertices. By Theorem 3, G has $2n - 2$ edges and every subgraph of the form $G - e$ has n vertices and $2n - 3$ generically independent edges. Thus $G - e$ is generically rigid, by the definition of infinitesimal rigidity in Section 2. Thus $\mathcal{G}_2$ satisfies C2, and hence C1, C3 and C4 by Lemma 7.

Suppose conversely that $\mathcal{G}$ is a non-empty class of graphs that is matroidal for dimension 2 and satisfies C1, C2, C3 or C4, and hence satisfies C4 by Lemma 7. We wish to prove that $\mathcal{G} = \mathcal{G}_2$. If $G \in \mathcal{G}$ then $G \in \mathcal{G}_2$ by Lemma 6(b) and Theorem 3. So suppose $G \in \mathcal{G}_2$. We shall prove by induction on $|E(G)|$ that $G \in \mathcal{G}$. Form a $\mathcal{G}$-closed graph G_c by adding edges to G as long as this does not increase the $\mathcal{G}$-rank, and let $F_1, \ldots, F_k$ be the maximal cliques of G_c. If $k = 1$ then G_c is complete and so, by Lemma 6(a) and Theorem 3,

$$r_{\mathcal{G}}(G) = r_{\mathcal{G}}(G_c) = 2|V(G)| - 3 \geqslant |E(G)| \geqslant r_{\mathcal{G}}(G).$$

Thus $r_{\mathcal{G}}(G) = |E(G)|$ and so $G \in \mathcal{G}$. So suppose $k \geqslant 2$ and let $G_i := G \cap F_i$ for each i. Then

$G_i \subsetneq G$ and so we may suppose inductively that $G_i \in \mathcal{G}$. By Lemma 6(c), $E(G_i) \cap E(G_j) = \varnothing$ whenever $i \neq j$. Thus, by C4,

$$r_{\mathcal{G}}(G) = r_{\mathcal{G}}(G_c) = \sum_i r_{\mathcal{G}}(F_i) \geqslant \sum_i |E(G_i)| = |E(G)| \geqslant r_{\mathcal{G}}(G).$$

Then $G \in \mathcal{G}$ as required. □

5. One-dimensional and three-dimensional frameworks.