On the Stretch Factor of the Constrained Delaunay Triangulation*

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Abstract

Given a set P of n points in the plane and a set S of non-crossing line segments whose endpoints are in P, let $CDT(P, S)$ be the constrained Delaunay triangulation of P with respect to S. Given any two visible points $p, q \in P$, we show that there exists a path from p to q in $CDT(P, S)$, denoted $SP_{CDT}(p, q)$, such that every edge in the path has length at most $|pq|$ and the ratio $|SP_{CDT}(p, q)|/|pq|$ is at most $\frac{4\pi\sqrt{3}}{9}(\approx 2.42)$, thereby improving on the previously known bound of $\frac{\pi(1+\sqrt{5})}{2}(\approx 5.08)$.

Key words: stretch factor, spanning ratio, spanner, constrained Delaunay triangulation, computational geometry.

1 Introduction

The spanning properties of various geometric graphs has been studied extensively in the literature (see Eppstein [4], Knauer and Gudmundsson [8], Narasimhan and Smid [10], Smid [11] for several surveys on the topic). In this article, we concentrate on the spanning ratio of the Delaunay triangulation in the constrained setting.

Before we can state our results precisely, we outline what we mean by the constrained setting and how the spanning ratio of a geometric graph is measured in this setting. We define a *geometric graph* to be a graph whose vertex set is a set of points in the plane, and whose edge set is a set of line segments joining pairs of vertices. Let P denote a set of n points in the plane. For simplicity, we assume that all point sets are in general position, i.e. no three points are collinear and no four are cocircular. Let S be a set of non-crossing line segments whose endpoints are in P. Two points p and q of P are *visible* with respect to S provided the segment pq does not properly intersect any segment of S. Two line segments intersect properly if they share a common interior point. The visibility graph of P constrained to S, denoted $Vis(P, S)$, is a geometric graph whose vertex set is P and whose edge set contains S as well as one edge for each visible pair of vertices. A spanning subgraph of $Vis(P, S)$ whose edge set contains S is a geometric graph *constrained to* S. In such a graph, the set S is referred to as the *constrained edges* and all other edges are referred to as *unconstrained edges* or *visibility edges*.

The graphs that we consider are weighted. The weight assigned to each edge $[pq]$ is the Euclidean distance between p and q. This weighting scheme allows us to define the notion of a constrained t-spanner.

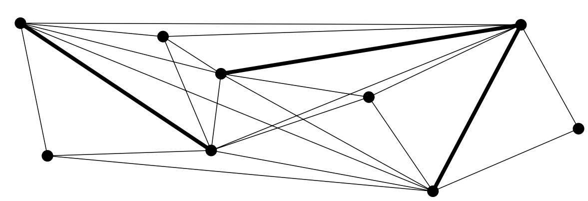

Figure 1. The visibility graph $Vis(P, S)$ where segments of S are shown in bold.

Definition 1.1 *Let $t \geq 1$ be a real number. A constrained geometric graph $G(P, S)$ is a* constrained t-spanner *provided that for every visibility edge $[pq]$ in $Vis(P, S)$, the length of the shortest path between p and q in $G(P, S)$ is at most t times the Euclidean distance between p and q. We refer to t as the* spanning ratio *or the* stretch factor *of $G(P, S)$.*

Note that if $G(P, S)$ is a constrained t-spanner, then for every pair of points p, q in P (not just visible edges), the

*Research supported in part by NSERC.

shortest path from p to q in $G(P, S)$ is at most t times the shortest path from p to q in $Vis(P, S)$.

The Delaunay triangulation of P constrained to S, denoted $CDT(P, S)$ is a triangulation of P where the edges have the following properties: (i) every segment in S is an edge of $CDT(P, S)$, and (ii) all visible pairs of points $p, q \in P$ form an edge of $CDT(P, S)$ provided that there exists a circle with p and q on its boundary that contains no other point of P visible from both p and q in its interior. The constrained Delaunay triangulation was first studied by Lee [9] who called it a generalized Delaunay triangulation. The term constrained Delaunay triangulation was coined by Chew [2]. Chew [2] and Wang and Schubert [12] independently showed that $CDT(P, S)$ can be computed in $O(n \log n)$ time.

Recently, several researchers noted that the technique of Dobkin *et al.* [3] used to prove a spanning ratio of $\frac{\pi(1+\sqrt{5})}{2} (\approx 5.08)$ for the Delaunay triangulation can be trivially extended to the constrained setting (see for example Klein *et al.* [7] or Karavelas [5]). Currently, the best known spanning ratio for the Delaunay triangulation is $\frac{2\pi}{3\cos(\pi/6)} = \frac{4\pi\sqrt{3}}{9} (\approx 2.42)$ by Keil and Gutwin [6]. The proof in Keil and Gutwin [6] cannot be trivially extended to the constrained setting. Our main result is the following:

Theorem 1.1 *Let P be a set of points in the plane and let S be a set of non-crossing line segments with endpoints in P. Let $[pq]$ be a visibility edge in $Vis(P, S)$. There is a path from p to q in $CDT(P, S)$ whose length is at most $\frac{4\pi\sqrt{3}}{9}|pq|$ and each edge in the path has length at most $|pq|$.*

In the unconstrained setting, the property of edge lengths in Theorem 1.1 was shown by Bose *et al.* [1]. This property has implications in the area of wireless networks. The standard graph used to model adhoc wireless networks is the unit disk graph (UDG). In the UDG, the vertex set is a set of points in the plane and two vertices are connected by an edge if the distance between two vertices is at most a given unit. The given unit usually represents the distance that wireless devices can transmit. The implication in the area of wireless networks is summarized in the following corollary. Given a constant $d > 0$ and a weighted graph G, let G_d denote the spanning subgraph with all edges of G whose weight is at most d.

Corollary 1.1 *Let P be a set of points in the plane and let S be a set of non-crossing line segments with endpoints in P. Let $d > 0$ be a given constant. Provided that $Vis_d(P, S)$ is connected, $CDT_d(P, S)$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of $Vis_d(P, S)$. That is, for every edge $[pq]$ in $Vis_d(P, S)$, there is a path from p to q in $CDT_d(P, S)$ whose length is at most $\frac{4\pi\sqrt{3}}{9}|pq|$.*

2 Spanning Ratio of the Constrained Delaunay Triangulation

In this section, we prove our main result. Our proof is by induction and we start by proving a lemma that is crucial for our inductive step. Before stating the lemma, we outline some of the terminology used. For the lemma below, all edges $[pq]$ will be directed edges from p to q.

Definition 2.1 *A circle with center m is a* right-empty *circle with respect to $[pq]$ provided that it has p and q on its boundary and no other point of P visible from both p and q in its interior to the right of $[pq]$. The* spanning-angle *of this right-empty circle, denoted $\theta(p, q)$, is the angle $\angle qmp$ (all angles are taken in counter-clockwise direction). The minimum spanning-angle of p and q with respect to P, denoted $\theta_M(p, q)$, is the smallest spanning-angle taken over all right-empty circles with respect to $[pq]$. The circle with minimum spanning-angle is denoted $RE(p, q)$. See Figure 2.*

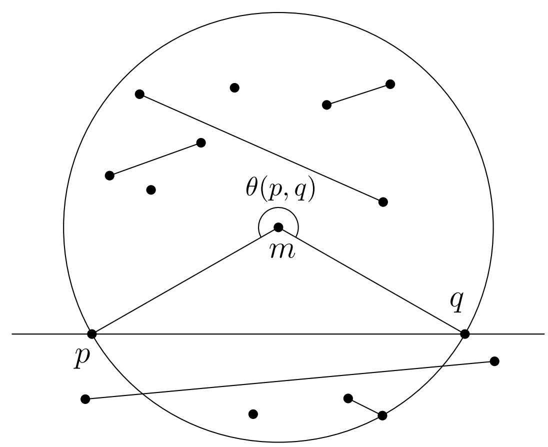

Figure 2. *Illustration of Definition 2.1.*

Lemma 2.1 *Let P be a set of n points in the plane, S be a set of non-crossing line segments with endpoints in P and $[pq]$ be a visibility edge in $Vis(P, S)$. If there is a right-empty circle of segment $[pq]$ with radius r, then there exists a path in $CDT(P, S)$ from p to q whose length is at most $r\theta(p, q)$ and every edge in that path has length at most $|pq|$.*

Proof: If $[pq]$ is an edge of the convex hull of P, then it is also an edge of $CDT(P, S)$, thus the lemma holds in this case. So for the remainder of the proof, we assume that $[pq]$ is not a convex hull edge.

Since there are $O(n^2)$ pairs of visible points, there are $O(n^2)$ minimum spanning-angles. We proceed by induction on the rank of the minimum spanning-angles (ties are broken arbitrarily).

Base Case: $\theta_M(p,q)$ has lowest rank. Let $RE(p,q)$ be the right-empty circle whose minimum spanning-angle is smallest over all minimum spanning-angles. We show that $[pq]$ is an edge of $CDT(P,S)$. Suppose that $[pq]$ is not an edge of $CDT(P,S)$. This implies that there must be at least one point of P in $RE(p,q)$ to the left of segment $[pq]$ that is visible to both p and q. Let $t \in P$ be such a point in $RE(p,q)$ such that the circle through p,t,q is $RE(q,p)$. Since there is at least one point to the left of $[pq]$ in $RE(p,q)$ visible to both p and q, the portion of $RE(q,p)$ to the right of $[qp]$ must be strictly contained in the portion of $RE(p,q)$ to the left of $[pq]$ (see Figure 3). This implies that no point of P visible to both p and q is contained strictly inside $RE(p,q) \cap RE(q,p)$.

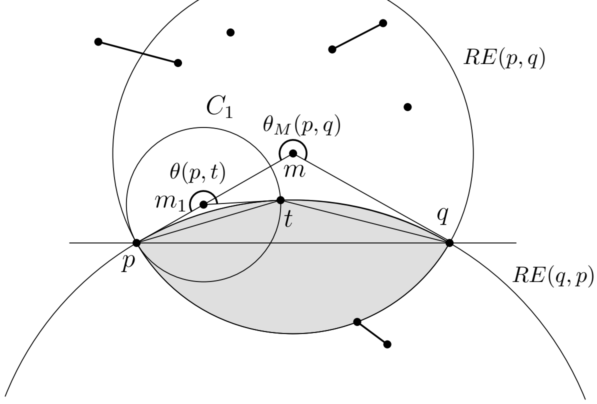

Figure 3. *Illustrating the base case ($RE(p,q) \cap RE(q,p)$ is highlighted).*

Consider the circle C_1 through p and t whose center lies on the segment pm. Since $RE(p,q) \cap RE(q,p)$ does not contain any points visible to both p and q, no point of P visible to both p and t lies in C_1 to the right of $[pt]$. Thus, C_1 is a right-empty circle with spanning angle $\theta(p,t)$. Now, since t is strictly to the left of $[pq]$ (otherwise we violate the fact that $[pq]$ is a visibility edge as well as the general position assumption), we have that $\theta(p,t) < \theta_M(p,q)$, which is a contradiction.

Inductive Hypothesis: For any visibility edge $[pq]$ whose minimum spanning-angle has rank at most $k \geq 1$, there is a path $SP_{CDT}(p,q)$ in $CDT(P,S)$ with length at most $r\theta_M(p,q)$ where every edge on that path has length at most $|pq|$ and r is the radius of a right-empty circle with respect to $[pq]$.

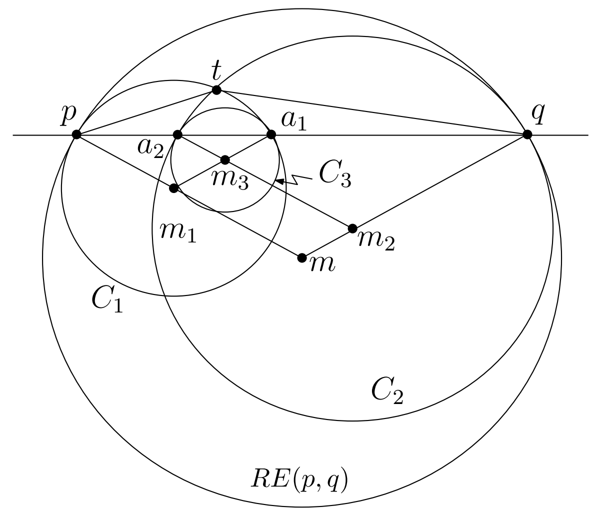

Figure 4. *Illustrating Case 1.*

Inductive Step: Consider a visibility edge $[pq]$ whose minimum spanning-angle has rank $k+1$. If $[pq]$ is an edge of $CDT(P,S)$, then we are done, therefore, assume that $[pq]$ is not an edge of $CDT(P,S)$. Let r be the radius of $RE(p,q)$. There must be at least one point of P in $RE(p,q)$ to the left of $[pq]$ that is visible to both p and q. As in the base case, let $t \in P$ be such a point in $RE(p,q)$ such that the circle through p,t,q is $RE(q,p)$.

Let C_1 be the circle through p and t whose center lies on the segment pm. We denote the center and radius of C_1 by m_1 and r_1, respectively. Similarly, let C_2 be the circle through t and q whose center lies on the segment qm. We denote the center and radius of C_2 by m_2 and r_2, respectively. By our choice of t, C_1 is a right-empty circle for $[pt]$ and we denote its spanning angle by $\theta_1(p,t)$. Similarly, C_2 is a right-empty circle for $[tq]$ with spanning angle $\theta_2(t,q)$.

Consider the two intersection points between C_1 and $[pq]$. One of these intersection points is p. Denote the other one by a_1. Similarly, let a_2 be the intersection point between C_2 and $[pq]$ that is not equal to q. If $|pa_1| > |pa_2|$ then the line through m_1 and a_1 and the line through m_2 and a_2 intersect at a point denoted m_3. Let C_3 be the circle through a_1 and a_2 centered at m_3. Denote the radius of C_3 by r_3.

Observe that the following four triangles are all similar isosceles triangles with two equal base angles, which we denote by ϕ: $\triangle(p,m,q)$, $\triangle(p,m_1,a_1)$, $\triangle(a_2,m_2,q)$, and

$\triangle(a_2, m_3, a_1)$.

We consider the cases based on the spanning angle $\theta_M(p,q)$ of $RE(p,q)$.

Case 1: $0 < \theta_M(p,q) \leq \pi$. (see Figure 4.)

In this case, notice that when the spanning angle is at most π, we have that $|pa_1| > |pa_2|$. As in the base case, note that both $\theta_1(p,t)$ and $\theta_2(t,q)$ are strictly less than $\theta_M(p,q)$ by construction. Similarly, the both $[pt]$ and $[tq]$ have length at most $[pq]$ since t lies in the circle with p and q as diameter. Therefore, by applying the inductive hypothesis on $[pt]$ and $[tq]$, we have that $SP_{CDT}(p,t) \leq r_1\theta_1(p,t)$ and $SP_{CDT}(t,q) \leq r_2\theta_2(t,q)$, respectively. Let $SP_{CDT}(p,q)$ be the concatenation of $SP_{CDT}(p,t)$ and $SP_{CDT}(t,q)$. Each edge on $SP_{CDT}(p,q)$ has length at most $|pq|$ by induction. We now bound its total length:

$$\begin{aligned} |SP_{CDT}(p,q)| &\leq |SP_{CDT}(p,t)| + |SP_{CDT}(t,q)| \\ &\leq r_1\theta_1(p,t) + r_2\theta_2(t,q) \\ &= r_1\theta_M(p,q) + r_2\theta_M(p,q) - \\ &\quad (r_1(\theta_M(p,q) - \theta_1(p,t)) + \\ &\quad r_2(\theta_M(p,q) - \theta_2(t,q))). \end{aligned}$$

Observe that

1. The length of the arc of C_3 from a_2 to a_1 in clockwise direction is equal to $r_3\theta_M(p,q)$ by construction.
2. The length of the arc of C_1 from t to a_1 in clockwise direction is equal to $r_1(\theta_M(p,q) - \theta_1(p,t))$.
3. The length of the arc of C_2 from a_2 to t in clockwise direction is equal to $r_2(\theta_M(p,q) - \theta_2(t,q))$.

Since C_3 is contained in $C_1 \cap C_2$, by convexity we have

$$\begin{aligned} r_1(\theta_M(p,q) - \theta_1(p,t)) &+ r_2(\theta_M(p,q) - \theta_2(t,q)) \\ &\geq r_3\theta_M(p,q) \end{aligned}$$

Hence,

$$\begin{aligned} |SP_{CDT}(p,q)| &\leq (r_1 + r_2 - r_3)\theta_M(p,q) \\ &= \left(\frac{|pa_1|}{2\cos\phi} + \frac{|a_2q|}{2\cos\phi} - \frac{|a_2a_1|}{2\cos\phi}\right)\theta_M(p,q) \\ &= \frac{|pq|}{2\cos\phi}\,\theta_M(p,q) \\ &= r\theta_M(p,q). \end{aligned}$$

Case 2: $\pi < \theta_M(p,q) < 2\pi$ and there is a point in $RE(p,q)$ to the left of $[pq]$ visible to both p and q that is also contained in the circle with p and q as diameter (see Figure 5. Circle with p and q as diameter is drawn with dashed boundary.)

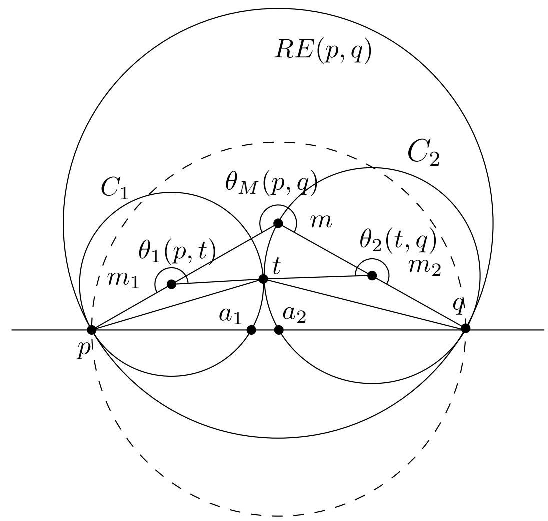

Figure 5. *Illustrating Case 2 with* $|pa_1| < |pa_2|$.

Suppose that $|pa_1| < |pa_2|$. We construct $SP_{CDT}(p,q)$ by concatenating $SP_{CDT}(p,t)$ with $SP_{CDT}(t,q)$. Since both $\theta_1(p,t)$ and $\theta_2(t,q)$ are strictly less than $\theta_M(p,q)$ by construction, we can apply the inductive hypothesis on $[pt]$ and $[tq]$.

Thus, we have

$$\begin{aligned} |SP_{CDT}(p,q)| &\leq |SP_{CDT}(p,t)| + |SP_{CDT}(t,q)| \\ &\leq r_1\theta_M(p,t) + r_2\theta_M(t,q) \\ &\leq (r_1 + r_2)\theta_M(p,q) \end{aligned}$$

Since $|pa_1| < |pa_2|$, we have that $[pa_1] \cap [a_2q] = \emptyset$. This means that $|pa_1| + |a_2q| < |pq|$. Thus, we have:

$$\begin{aligned} |SP_{CDT}(p,q)| &\leq (r_1 + r_2)\theta_M(p,q) \\ &= \left(\frac{|pa_1|}{2\cos\phi} + \frac{|a_2q|}{2\cos\phi}\right)\theta_M(p,q) \\ &\leq \frac{|pq|}{2\cos\phi}\theta_M(p,q) \\ &= r\theta_M(p,q). \end{aligned}$$

If $|pa_1| > |pa_2|$ then the argument given in Case 1 applies.

Finally, since t is inside the circle with p and q as diameter, we have that all edges in $SP_{CDT}(p,q)$ have length at most $|pq|$.

Case 3: $\pi < \theta_M(p,q) < 2\pi$ and there is no point in $RE(p,q)$ to the left of $[pq]$ visible to both p and q that is contained in the circle with p and q as diameter.

Since the circle with p and q as diameter does not contain any points to the left of $[pq]$, this means that $\theta_M(q,p) < \pi$. This implies that the rank of $\theta_M(q,p)$ is less than the rank of $\theta_M(p,q)$. Since the radius of $RE(q,p)$ is smaller than r (the radius of $RE(p,q)$), by the inductive hypothesis, we have the following:

$$\begin{aligned} |SP_{CDT}(p,q)| = |SP_{CDT}(q,p)| &\leq r\theta_M(q,p) \\ &\leq r\theta_M(p,q) \end{aligned}$$

■

We now use Lemma 2.1 to prove our main result.

Proof of Theorem 1.1:

Without loss of generality assume that p and q lie on a horizontal line L with p left of q. Let s be a point on L to the right of p. Let $C_1^s(C_2^s)$ be the circle that passes through p and s with center $o_1^s(o_2^s)$ above (below) L such that the angles $\angle spo_1^s(\angle o_2^s ps)$ and $\angle o_1^s sp(\angle pso_2^s)$ equal $\frac{\pi}{6}$. Let $Lune(p,s)$ be the intersection of the interiors of C_1^s and C_2^s. See Figure 6. If no points of P visible from p lie inside $Lune(p,q)$, then C_1^q is empty below L. Therefore, C_1^q is a right-empty circle and Lemma 2.1 implies that $SP_{CDT}(p,q) \leq r_1^q \theta_1^q$ where r_1^q is the radius of C_1^q and θ_1^q is the spanning-angle. We have $\theta_1^q = 2\pi - (\pi - 2(\frac{\pi}{6})) = \frac{4\pi}{3}$ and $r_1^q = \frac{|pq|}{2cos(\frac{\pi}{6})}$, thus $SP_{CDT}(p,q) \leq \frac{4\pi\sqrt{3}}{9}|pq|$. Also, Lemma 2.1 guarantees that all edges in $SP_{CDT}(p,q)$ have length at most $|pq|$.

It remains to consider the case where there exists a point of P inside $Lune(p,q)$ that is visible from p. Fix s to be the rightmost point such that $Lune(p,s)$ contains no point of P visible from p. Thus there exists a point t of P on the boundary of $Lune(p,s)$ that is visible from p and $Lune(p,s)$ is contained in $Lune(p,q)$. Without loss of generality we may assume that t lies above L. See Figure 7. Let θ be angle $\angle spt$. Let C^t be the circle through p and t whose center o^t lies on the line through p and o_1^s. By construction, C^t is a right-empty circle for $[pt]$. Let $r = |po^t|$ be the radius of C^t and let α be its spanning-angle.

If t lies below the line through p and o^t, then as angle $\angle spo^t$ is $\frac{\pi}{6}$ we know that angle $\angle tpo^t$ is $\frac{\pi}{6} - \theta$, thus the lower angle $\angle po^t t$ inside the triangle $\Delta po^t t$ is $\pi - 2(\frac{\pi}{6} - \theta) = \frac{2\pi}{3} + 2\theta$ and $\alpha = 2\pi - (\frac{2\pi}{3} + 2\theta) = \frac{4\pi}{3} - 2\theta$. Similar calculations show that α remains equal to $\frac{4\pi}{3} - 2\theta$ if t lies on or above the line through p and o^t.

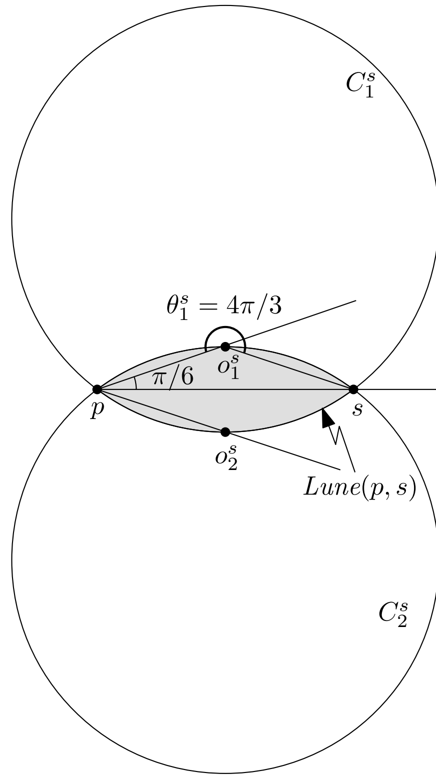

Figure 6. Definition of $Lune(p,s)$

To determine a value for r we again first assume that t lies below the line through p and o^t. We begin by using the sine law to determine $|pt|$. Since t is on $Lune(p,s)$ we have that angle $\angle pts$ is $\frac{2\pi}{3}$. Thus $\frac{|pt|}{sin(\frac{\pi}{3}-\theta)} = \frac{|ps|}{sin(\frac{2\pi}{3})}$. Thus $|pt| = |ps|\frac{sin(\frac{\pi}{3}-\theta)}{sin(\frac{2\pi}{3})}$. Again using the sine law

$$\frac{|pt|}{sin(\frac{2\pi}{3} + 2\theta)} = \frac{r}{sin(\frac{\pi}{6} - \theta)}$$

and

$$\begin{aligned} r &= |pt|\frac{sin(\frac{\pi}{6} - \theta)}{sin(\frac{2\pi}{3} + 2\theta)} \\ &= |ps|\frac{sin(\frac{\pi}{3} - \theta)sin(\frac{\pi}{6} - \theta)}{sin(\frac{2\pi}{3})2sin(\frac{\pi}{3} + \theta)cos(\frac{\pi}{3} + \theta)} \\ &= |ps|\frac{sin(\frac{\pi}{3} - \theta)}{2sin(\frac{2\pi}{3})sin(\frac{\pi}{3} + \theta)} \end{aligned}$$

Again the value of r does not change if t lies on or above the line through p and o^t.

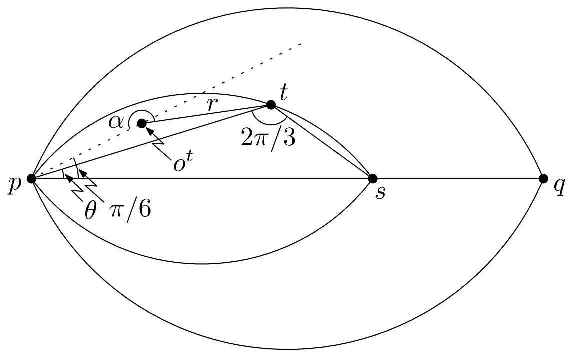

Figure 7. t is visible from p

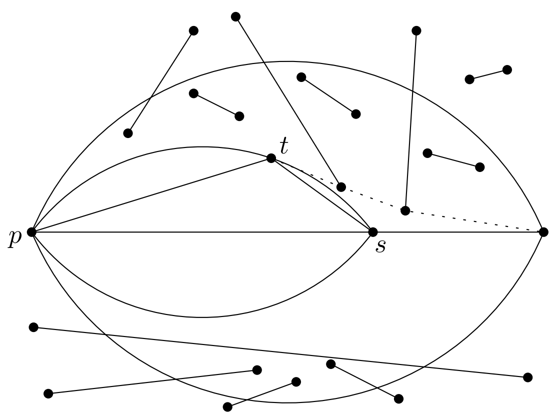

Figure 8. The path from t to q in $Vis(P, S)$ is no longer than $|ts| + |sq|$

We now proceed to prove the theorem by induction on the rank of $|pq|$ amongst visibility edges in P. If p and q are the two closest points in P, the $Lune(p, q)$ is empty and by Lemma 2.1 $SP_{CDT}(p, q) \le \frac{4\pi\sqrt{3}}{9}$ and each edge on $SP_{CDT}(p, q)$ has length at most $|pq|$.

As an inductive step consider the ith closest visible pair p and q and assume that for all closer pairs the theorem holds. We know

$$SP_{CDT}(p, q) \le SP_{CDT}(p, t) + SP_{CDT}(t, q)$$

We know that circle C^t through p and t is empty of points visible from p below the line through p and t. Thus by Lemma 2.1

$$SP_{CDT}(p, t) \le \alpha r = |ps| \frac{(\frac{2\pi}{3} - \theta) sin(\frac{\pi}{3} - \theta)}{sin(\frac{2\pi}{3}) sin(\frac{\pi}{3} + \theta)}$$

It remains to bound $SP_{CDT}(t, q)$. Since $Lune(p, s)$ is empty of points visible from p and since q is visible from p, no edge of S intersects segment ts. Also since p is visible from q, no edge of S intersects segment sq. Thus either t is visible to q or there exists a path from t to q along the boundary of the convex hull of the points of P in triangle Δtsq whose length is less than $|ts| + |sq|$ by convexity. We have $|sq| = |pq| - |ps|$ and by the sine law $|ts| = |ps| \frac{sin(\theta)}{sin(\frac{2\pi}{3})}$.

To prove the theorem we must show that

$$SP_{CDT}(p, q) \le SP_{CDT}(p, t) + SP_{CDT}(t, q) \le \frac{4\pi\sqrt{3}}{9}|pq|$$

In particular, using the inductive assumption on the path from t to q we must show that

$$SP_{CDT}(p, q) \le \alpha r + \frac{4\pi\sqrt{3}}{9}[|ts| + |sq|] \le \frac{4\pi\sqrt{3}}{9}|pq|$$

Thus it suffices to show that

$$\alpha r \le \frac{4\pi\sqrt{3}}{9}(|pq| - |ts| - |sq|)$$

or

$$\frac{\alpha r}{|pq| - |ts| - |sq|} \le \frac{4\pi\sqrt{3}}{9}$$

The left hand side of this inequality is equal to

$$\begin{aligned}
&\frac{\alpha r}{|pq| - |ts| - |sq|} \\
= \quad & \frac{|ps|(\frac{2\pi}{3} - \theta) sin(\frac{\pi}{3} - \theta)}{sin(\frac{2\pi}{3}) sin(\frac{\pi}{3} + \theta)(|ps| + \frac{|ps| sin(\theta)}{sin(\frac{2\pi}{3})})} \\
= \quad & \frac{(\frac{2\pi}{3} - \theta) sin(\frac{\pi}{3} - \theta)}{sin(\frac{2\pi}{3}) sin(\frac{\pi}{3} + \theta)(1 - \frac{sin(\theta)}{sin(\frac{2\pi}{3})})} \\
= \quad & \frac{(\frac{2\pi}{3} - \theta) sin(\frac{\pi}{3} - \theta)}{sin(\frac{\pi}{3} + \theta)(sin(\frac{2\pi}{3}) - sin(\theta))} \\
\stackrel{\text{def}}{=} \quad & f(\theta)
\end{aligned}$$

It remains to show that $f(\theta) \le \frac{4\pi\sqrt{3}}{9}$ when $0 \le \theta < \frac{\pi}{3}$. Inside the range $0 \le \theta < \frac{\pi}{3}$, $f(\theta)$ has one local minimum and no local maxima, thus $f(\theta)$ achieves its largest value when $\theta = 0$ or when θ approaches $\frac{\pi}{3}$. If $\theta = 0$,

$$f(\theta) = \frac{\frac{2\pi}{3} sin(\frac{\pi}{3})}{sin(\frac{\pi}{3}) sin(\frac{2\pi}{3})} = \frac{2\pi}{3 sin(\frac{2\pi}{3})} = \frac{4\pi\sqrt{3}}{9}$$

To evaluate $\lim_{\theta \to \frac{\pi}{3}} f(\theta)$ we use l'Hopital's rule.

$$f(\theta) = \frac{(\frac{2\pi}{3} - \theta)\sin(\frac{\pi}{3} - \theta)}{\sin(\frac{\pi}{3} + \theta)(\sin(\frac{2\pi}{3}) - \sin\theta)} \stackrel{\text{def}}{=} \frac{g(\theta)}{h(\theta)}$$

With this definition, we can evaluate the limit as θ approaches $\frac{\pi}{3}$.

$$\lim_{\theta \to \frac{\pi}{3}} f(\theta) = \lim_{\theta \to \frac{\pi}{3}} \frac{g(\theta)}{h(\theta)} \quad = \quad \lim_{\theta \to \frac{\pi}{3}} \frac{g'(\theta)}{h'(\theta)}$$

$$= \quad \frac{-(\frac{\pi}{3})\cos(0) - \sin(0)}{-\sin(\frac{2\pi}{3})\cos\frac{\pi}{3} + (\sin(\frac{2\pi}{3}) - \sin\frac{\pi}{3})\cos(\frac{2\pi}{3})}$$

$$= \quad \frac{-\frac{\pi}{3}}{-\frac{1}{2}\sin(\frac{2\pi}{3})} = \frac{2\pi}{3\sin\frac{\pi}{3}} = \frac{4\pi\sqrt{3}}{9}$$

Since neither $[pt]$ nor $[tq]$ is longer than $[pq]$, each edge in $SP_{CDT}(p,t)$ and $SP_{CDT}(t,q)$ has size at most $|pq|$ by induction. We conclude that every edge on $SP_{CDT}(p,q)$ has length at most $|pq|$.

■

3 Conclusions

Given a set P of points in the plane and a set S of non-crossing line segments with endpoints in P, we have shown that there is a path from p to q in $CDT(P,S)$ (where $[pq]$ is an edge in $Vis(P,S)$) whose length is at most $\frac{4\pi\sqrt{3}}{9}|pq|$ and each edge in the path has length at most $|pq|$.

By putting points close to the boundary of a circle, one can show that the lower bound on the spanning ratio can approach $\frac{\pi}{2}$. Closing the gap between our upper bound of $\frac{4\pi\sqrt{3}}{9}$ and the lower bound of $\frac{\pi}{2}$ remains the main open problem in this area.

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