Stable and Topology-Preserving Extraction of Medial Axes

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Abstract

The paper presents a simple method for extracting global medial axes in a stable manner. A new index, called the normalized boundary distance, is introduced in order to measure the degree of importance of a point on the conventional medial axis. This index has a remarkable property that the set of points whose index values are greater than an arbitrarily chosen threshold is topologically equivalent to the original figure. In the proposed method, first the boundary of a given figure is replaced with a dense set of points, next the Voronoi diagram for these points is constructed, then the approximation of the medial axis is extracted from the Voronoi diagram, and finally the global medial axis is constructed by pruning the branches according to the new index. The performance of the proposed method is also shown by examples.

1. Introduction

The medial axis is one of the most basic structures for representing features of two-dimensional figures [6], but it is not easy to extract the medial axis in a stable manner. The main reason for the difficulty comes from a gap between what we want to mean by the "medial axis" and what is actually defined in a mathematical way.

Conceptually, we want to mean by the "medial axis" a skeletal structure that represents a global feature of a figure just like the skeletons of a human body and the stroke information of hand-written character images. Therefore, the medial axis must not be disturbed by small noises of the figure.

Mathematically, on the other hand, the medial axis is defined as the set of points that are inside the figure and that are the centers of circles touching the boundary of the figure at two or more points. Therefore, the medial axis according to this definition is sensitive to small change of the boundary of the figure. Hence we are interested in extracting only a global structure of the medial axis that are independent from small noises of the boundary of the figure.

There have been many attempts to extract a global structure of the medial axes stably. The basic methods are classified into three groups: the use of thinning operations for digital pictures [10, 11, 14], the use of finite difference equations for simulation of fire expansion [12, 17], and the use of Voronoi diagram [1, 4, 15]. These methods are combined with techniques for improving the stability. They include the smoothing techniques [5] and more complicated processing [3, 9, 16, 18], which we will review in the next section.

However, there is no perfect method. The previous methods require either high computational cost for smoothing or complicated structures of processing.

This paper presents a new method for extracting a global structure of the medial axes quickly and stably. We introduce an index of strength for each point on the mathematically defined medial axis. The proposed index has the following good properties. First, a larger value of this index implies that the point represents a more global nature of the figure. Secondly, for any threshold value, if we collect all the points whose indices are greater than the threshold, we get a skeleton which is topologically equivalent to the original figure. Therefore, our method does not require any preprocessing for smoothing, but still can extract the global structure of the medial axis in a stable way.

The organization of the paper is as follows. In section

2 we will see how unstable the mathematically defined medial axis is, and then will briefly review existing methods for medial axis extraction. In section 3 we will define an index of strength, which we call a "normalized boundary distance", and in section 4 we will prove basic properties of the normalized boundary distance. In section 5, we will propose a new method for extracting the medial axis for a simply connected figure, and in section 6 we will extend the measure as well as the algorithm for general figures. Finally in section 7, we will give a concluding remark together with future work.

2. Instability of the Medial Axis and a Global Medial Axis

Let A be a bounded closed set of points in the plane. For a while we assume that A is simply connected (we will remove this assumption later). We will call A a *figure*. The medial axis of the figure A is defined as the set of centers of circles that are inside A and that touch the boundary of A at two or more points. More formally, we can express this definition in the following way.

Let us denote the boundary of A by ∂A. Let $c(x, y; r)$ denote the circle centered at (x, y) with radius r. For set X, we denote the number of elements of X by $|X|$. We define $M(A)$ as

$$M(A) = \{(x, y) \mid c(x, y; r) \subset A,\ |c(x, y; r) \cap \partial A| \geq 2\}, \tag{1}$$

and call $M(A)$ the *medial axis* of A. For each point $(x, y) \in M(A)$, the circle satisfying $c(x, y; r) \subset A$ and $|c(x, y; r) \cap \partial A| \geq 2$ is called the *touching circle* centered at (x, y).

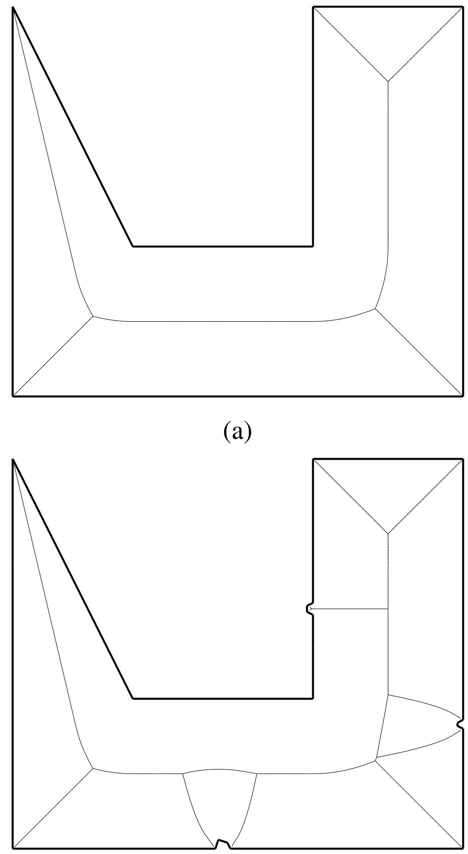
Fig. 1(a) shows an example of the medial axis; the thick lines represent the boundary of a figure wheares the thin lines represent the medial axis. In this example, the medial axis represents the global structure of the figure in a simple manner.

Fig. 1(b), on the other hand, shows how unstable the medial axis is. The figure in Fig. 1(b) has small disturbances on its boundary, and the associated medial axis has many branches due to these disturbances. As shown in this example, the medial axis is sensitive to small deviation of the boundary of the figure.

What we want to extract is a basic part of the medial axis that represents a global structure of the figure. It is actually an important open problem to find that kind of a basic part of the medial axis in a stable way.

In this paper, let us call this basic part of the medial axis the "global medial axis". Note that the term "global medial axis" is not defined clearly; instead it just represents what we want to extract from the figure.

However, we can say that the global medial axis should satisfy at least the following properties.



(a)

(b)

Figure 1. Medial axis: (a) simple figure; (b) figure with noises on the boundary.

Property 1. It should not be affected by small disturbance of the boundary of the figure.

Property 2. It should be invariant under scale transformation, i.e., expansion and shrinkage.

Property 3. It should be invariant under translations and rotations of the figures.

Property 4. It should be connected if the figure is connected.

Property 5. It should have a hierarchical structure in the sense that as we neglect small deviation of the boundary more and more, the associated global medial axis becomes more and more simple.

Many methods have already been proposed in order to attain these properties. The following are typical methods.

(A) Smoothing of the boundary [5]

In this method, the original figure A is changed to another figure A' by removing high frequency components of the boundary of A using low-pass filters, and then the medial axis $M(A')$ of A' is constructed. This method satisfies Properties 1, 2, 3 and 4, but does not satisfy Property 5. Actually this method can generate new branches that do not exist in the original medial axis [18]. Moreover, in order to obtain the hierarchical structure, we have to apply low-pass filters with different sizes repeatedly, and hence have to pay high computational cost.

(B) Pruning by shock-wave velocity [6, 14]

In this method, they suppose that wave propagates from the boundary towards inside of the figure in a constant speed. Then, two different wave fronts meet at the medial axis, and generate shock wave. The speed of the shock wave can be regarded as the degree of importance of the medial axis. Hence, they collect shock waves with high speed and regard it as a global medial axis. However, in this method, the extracted structure does not satisfy Property 4. In order to guarantee the connectedness, they require complicated pre- and post-processing.

(C) Pruning by touching-circle radii [4, 9]

For each branch of the medial axis, the maximum radius of the touching circles centered on this branch is considered as the degree of importance of the branch. We can prune the branches according to this degree of importance. Branches caused by small disturbance of the boundary curve can actually been removed by this pruning. However, the structure extracted by this method does not satisfy Property 4; if a figure includes a bottle-neck shape, the pruned axis becomes disconnected. Hence, this method also requires complicated post-processing to guarantee the connectedness.

(D) Pruning by the boundary-radius ratio [3, 15]

In this method, the ratio of the radius of the touching circle to the length of the boundary curve between the two contact points is considered as the degree of importance of the point on the medial axis, and the medial axis is pruned by this ratio. The resulting structure does not satisfy Property 4, either, and hence requires heuristic post-processing.

(E) Pruning by the distance from the smallest touching circle [18]

For each point on the medial axis, the distance from the smallest touching circle on the same branch is considered as the degree of importance of that point, and the medial axis is pruned. The resulting structure of this method does not satisfy Property 4, and hence heuristic post-processing is required, too.

As we have seen above, many methods have been proposed for trying to find the global medial axis. However, they require high computational cost in pre-processing for smoothing in various scales, or requires complicated heuristic post-processing for guaranteeing the connectedness of the resulting structure.

In the next section we will define another index to measure the degree of importance of a point on the medial axis. This index is remarkable in the sense that pruning the medial axis by this index results in the structure that satisfies all the five properties; in particular it guarantees the connectedness of the structure. Therefore, pruning by this index, we can extract desired global medial axis without expensive pre-processing or heuristic post-processing.

3. Normalized Boundary Distance

Let A be a figure. Since A is assumed to be simply connected, the boundary ∂A of A forms a single closed curve. Let us denote the length of this boundary by L.

Let P be an arbitrary point on the medial axis $M(A)$, and let $c(\mathrm{P})$ be the touching circle centered at P. From the definition of the medial axis, we have $|c(\mathrm{P}) \cap \partial A| \geq 2$, and hence the boundary ∂A is partitioned into two or more pieces by the contact points, i.e., the points in $c(\mathrm{P}) \cap \partial A$. Among them, the second longest piece is called the *support* of P.

In most cases the touching circle $c(\mathrm{P})$ has exactly two contact points. In those cases the support means the shorter boundary path between the two contact points.

At some point $\mathrm{P} \in M(A)$, on the other hand, three or more branches of the medial axis meet together. Then, the touching circle $c(\mathrm{P})$ has three or more contact points, and the boundary of A is partitioned into the same number of pieces as the number of branches incident to P. The support of P corresponds to the second largest boundary piece instead of the shortest piece.

Let $l(\mathrm{P})$ be the length of the support of P. We define $\beta(\mathrm{P})$ by

$$\beta(\mathrm{P}) = \frac{2l(\mathrm{P})}{L}, \tag{2}$$

and call $\beta(\mathrm{P})$ the *normalized boundary distance*. Note that $l(\mathrm{P}) \leq L/2$, and hence we have

$$0 < \beta(\mathrm{P}) \leq 1. \tag{3}$$

We consider that $\beta(\mathrm{P})$ represents the degree of importance of the point P, and prune the medial axis according to $\beta(\mathrm{P})$. In other words, for any T such that $0 \leq T \leq 1$, we define

$$M_T(A) = \{\mathrm{P} \mid \mathrm{P} \in M(A), \beta(\mathrm{P}) \geq T\}, \tag{4}$$

and call $M_T(A)$ the *global medial axis* with respect to the threshold T.

4. Monotone Property of the Normalized Boundary Distance

Since the figure A is simply connected, the associated medial axis $M(A)$ is a tree in a graph-theoretic sense. What we want to guarantee is that for any T such that $0 \le T \le 1$, the associated global medial axis $M_T(A)$ is also a tree. Actually we can prove the next theorem.

Theorem 1. If A is simply connected, $M_T(A)$ is a tree in a graph-theoretic sense for any T such that $0 \le T \le 1$.

To prove this theorem, we need some preparation. Suppose that $\mathrm{P} \in M(A)$, and that Q is one of the contact points of the touching circle $c(\mathrm{P})$. The line segment $\overline{\mathrm{PQ}}$ is called a *leg* of P, and Q is called a *foot* of P. ! ! We denote by $d(\mathrm{P},\mathrm{Q})$ the Euclidean distance between two points P and Q.

Lemma 1. Suppose that $\mathrm{P},\mathrm{P}' \in M(A)$ are two distinct points. Let Q be a foot of P, and Q$'$ be a foot of P$'$. If $\mathrm{Q} \ne \mathrm{Q}'$, the two legs $\overline{\mathrm{PQ}}$ and $\overline{\mathrm{P'Q'}}$ do not intersect.

Proof. Assume that $\overline{\mathrm{PQ}}$ and $\overline{\mathrm{P'Q'}}$ intersect. Since the four points P, P$'$, Q, Q$'$ are mutually distinct, they form a convex quadrilateral, and $\overline{\mathrm{PQ}}$ and $\overline{\mathrm{P'Q'}}$ are its diagonals. Hence we have

$$d(\mathrm{P},\mathrm{Q}) + d(\mathrm{P}',\mathrm{Q}') > d(\mathrm{P},\mathrm{Q}') + d(\mathrm{P}',\mathrm{Q}). \quad (5)$$

On the other hand, the touching circle $c(\mathrm{P})$ is contained in A and Q$'$ is on the boundary of A. Hence

$$d(\mathrm{P},\mathrm{Q}) \le d(\mathrm{P},\mathrm{Q}'). \quad (6)$$

Similarly $c(\mathrm{P}')$ is contained in A and Q is on the boundary of A, and consequently

$$d(\mathrm{P}',\mathrm{Q}') \le d(\mathrm{P}',\mathrm{Q}). \quad (7)$$

The inequalities (6) and (7) contradict (5). Thus we have Lemma 1. ∎

Let $\mathrm{P}^* \in M(A)$ be the point that attains the maximum of the normalized boundary distance, i.e.,

$$\beta(\mathrm{P}^*) = \max_{\mathrm{P} \in M(A)} \beta(\mathrm{P}). \quad (8)$$

We call P^* the *center* of the medial axis $M(A)$.

The medial axis $M(A)$ forms a tree consisting of a finite number of edges. In general P^* is an interior point of an edge, and in that case we have $\beta(\mathrm{P}^*) = 1$. On the other hand, if P^* is a vertex incident to three or more edges, we have $\beta(\mathrm{P}^*) < 1$.

In what follows, we consider the medial axis $M(A)$ as the rooted tree with the root P^*. This means that if P^* is an interior point of an edge, we divide the edge at P^* into two edges and consider that these two edges are incident to the root P^*. The vertices other than the root have either one edge or more than two edges. A vertex with exactly one edge is called a *leaf vertex*, and a vertex with more than two edges is called a *branching vertex*.

Lemma 2. Let P^* be the center of the medial axis $M(A)$ and Q be a leaf vertex. Let $\mathrm{P}_1 = \mathrm{P}^*, \mathrm{P}_2, \mathrm{P}_3, \ldots, \mathrm{P}_k = \mathrm{Q}$ be the list of all the vertices that we encounter in this order when we travel along the path from the root P^* to the leaf vertex Q.

Let R, R^* be interior points on an edge connecting P_i and P_{i+1} $(1 \le i \le k-1)$ which we encounter in this order when we move from P^* to Q. Then we have $\beta(\mathrm{R}) \ge \beta(\mathrm{R}')$.

Proof. Since R and R$'$ are interior points of the same edge, they have exactly two legs and they do not intersect. Since R$'$ is farther than R from the center along the path, the support of R$'$ is contained in the support of R. Hence we have $\beta(\mathrm{R}) \ge \beta(\mathrm{R}')$. ∎

The property essentially equivalent to Lemma 2 have already been observed in some previous work [3, 15]. However, as we reviewed in section 2 (D), they modified the index in other directions and did not reach Theorem 1.

Lemma 3. Let $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_k$ be the same as in Lemma 2. When point $\mathrm{P} \in M(A)$ passes through P_i on its way from P^* $(= \mathrm{P}_1)$ to Q $(= \mathrm{P}_k)$, the normalized boundary distance $\beta(\mathrm{P})$ decreases.

Proof. Let s_1 be the support of P just before P passes through P_i, and s_2 be the support of P immediately after P passes through P_i. Case 1. Suppose that s_2 is contained in the support of P_i. Then, s_2 is also contained in s_1 and consequently, $\beta(\mathrm{P})$ decreases continuously when P passes through P_i. Case 2. Suppose that s_2 is not contained in the support of P_i. Then, the length of s_2 is smaller than the length $l(\mathrm{P}_i)$ of the support of P_i, and consequently $\beta(\mathrm{P})$ decreases when P passes through P_i although the change of $\beta(\mathrm{P})$ may be discontinuous. Therefore, in both cases $\beta(\mathrm{P})$ decreases when it passes through P_i. ∎

From Lemmae 2 and 3, we can see that when we move from the center P^* towards a leaf vertex along the medial axis $M(A)$, the normalized boundary distance decreases monotonically. Hence we get Theorem 1.

5. Global Medial Axis Based on the Normalized Boundary Distances

In this section, we construct a method for extracting the global medial axis using the normalized boundary distance. For that purpose, we first construct the Voronoi diagram, next extract an approximation of the medial axis, and finally obtain the global medial axis by pruning the branches according to the normalized boundary distances.

5.1. Voronoi Diagram as an Approximation of the Medial Axis

Let $S = \{\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n\}$ be a set of n points in the plane $\mathbf{R}^2$. By $R(S; \mathrm{P}_i)$ we denote the set of points that are nearer to P_i than the any other point in S. In other words, we define

$$R(S; \mathrm{P}_i) = \{\mathrm{P} \in \mathbf{R}^2 \mid d(\mathrm{P}, \mathrm{P}_i) < d(\mathrm{P}, \mathrm{P}_j),\ j \neq i\}. \quad (9)$$

$R(S; \mathrm{P}_i)$ is called the *Voronoi region* of P_i. The plane is partitioned into $R(S; \mathrm{P}_1), R(S; \mathrm{P}_2), \ldots, R(S; \mathrm{P}_n)$ and their boundaries. This partition is called the Voronoi diagram for S [20]. The points in S are called *generators* of the Voronoi diagram. Fig. 2 shows an example of the Voronoi diagram. Line segments or half lines shared by the boundaries of two Voronoi regions are called *Voronoi edges*, and the points at which the boundaries of three or more Voronoi regions meet are called *Voronoi vertices*.

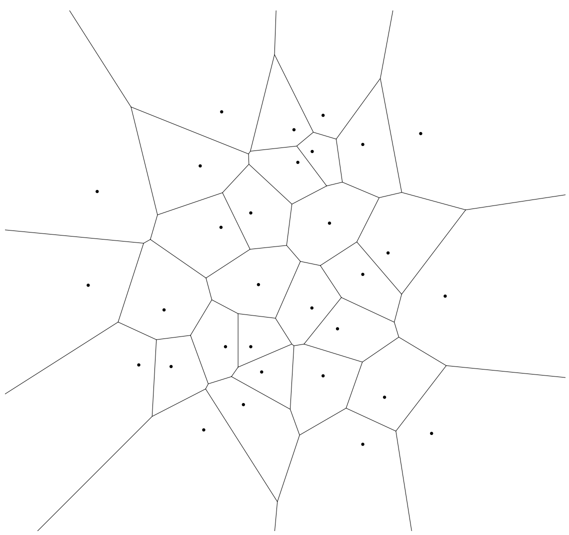
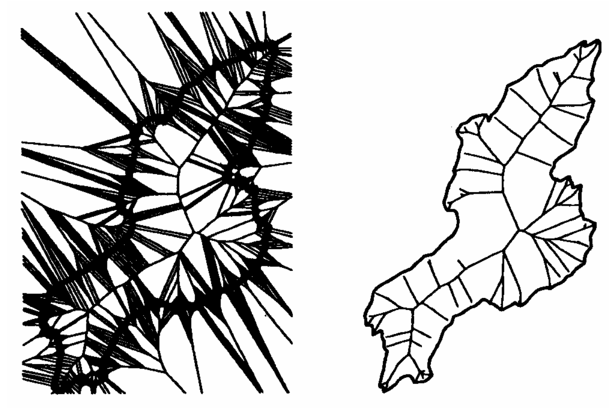


Figure 2. Voronoi diagram.

The Voronoi edge is in equal distance from the associated two generators. Using this property, we can extract an approximation of the medial axis of a given figure A in the following manner. As shown in Fig. 3(a), we first replace the boundary of A with a sequence of points densely placed on the boundary, and then construct the Voronoi diagram. From this Voronoi diagram we remove those Voronoi edges which are not contained in A. Then we get a subgraph of the Voronoi diagram, as shown in Fig. 3(b). This subgraph can be considered as an approximation of the medial axis, because for any point Q on the remaining Voronoi edges, the circle centered at Q with the radius $d(\mathrm{Q}, \mathrm{P}_i)$ for the generator P_i in either side of the Voronoi edge does not contain any generator in its interior (meaning that the circle is contained in the figure A), and hence the circle can be considered as the touching circle and the two associated generators can be considered as the feet of Q; hence Q is considered to be on the medial axis.



(a) (b)

Figure 3. Example 1: (a) Voronoi diagram; (b) Voronoi edges inside the boundary (primary medial axis).

5.2. Extraction of a Global Medial Axis

On the basis of our observations, we can construct the following algorithm for extracting a global medial axis.

Algorithm 1 (global medial axis).
Input: Simply connected figure A in the plane and threshold T.
Output: Global medial axis $M_T(A)$.
Procedure:
1. Place generators densely on the boundary of A.
2. Construct the Voronoi diagram for the generators.
3. Remove from the Voronoi diagram those edges that are not contained in A. [Comment: the resulting subgraph is considered as the medial axis $M(A)$.]
4. Prune branches by removing the Voronoi edges whose normalized boundary distances are smaller than T, and report the remaining structure. ∎

Note that the normalized boundary distance $\beta(\mathrm{P})$ is originally defined for point P on the medial axis, whereas in Step 4 of the algorithm, the normalized boundary distance is computed for each edge. This is not inconsistent because of the following reason. Let e be an edge in the subgraph obtained in Step 3 of the algorithm, and let P_i and P_j be the associated two generators. Then, P_i and P_j can be considered as feet of any point P on e, and the shorter boundary path connecting P_i and P_j can be considered as the support of P. Therefore, the normalized boundary distance $\beta(\mathrm{P})$ is constant on the edge.

Fig. 4 shows the global medial axes for the figure in Fig. 3 obtained by our algorithm. In this figure, (a), (b), (c) and (d) correspond to $M_T(A)$ for $T = 0.03, 0.06, 0.12$ and 0.24 respectively. As guaranteed by Theorem 1, $M_T(A)$'s are trees for any T.

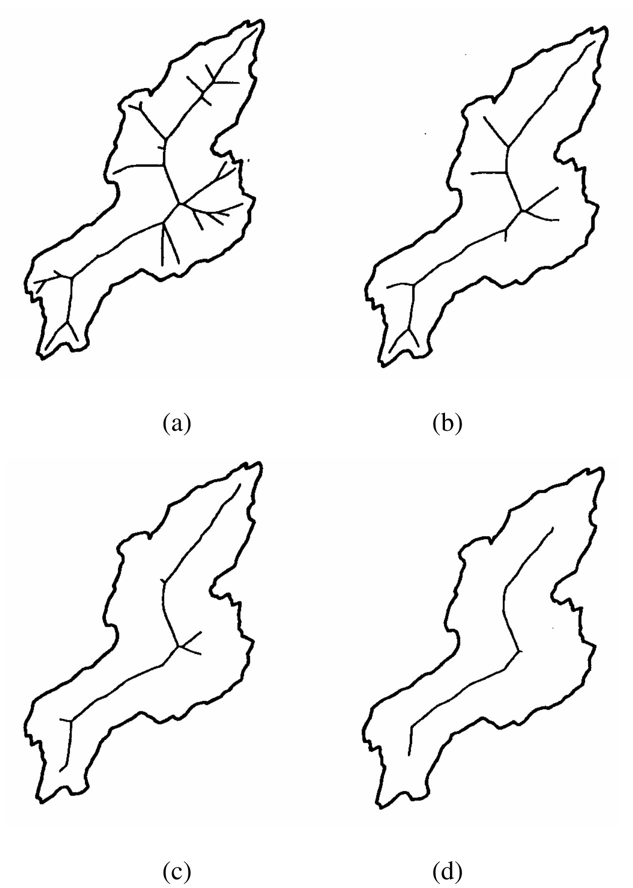


Figure 4. Example 1 (continued): (a) Edges with the index greater than 0.03; (b) Edges with the index greater than 0.06; (c) Edges with the index greater than 0.12; (d) Edges with the index greater than 0.24.

6. Extension to General Figures

Until now, we considered only simply connected figures. From now on we remove this restriction and consider figures in general, i.e., figures consisting of one or more connected components with possible holes.

Let A be a figure, and $M(A)$ be the medial axis of A. Since A may have two or more connected components and/or may have holes, the boundary ∂A is a collection of one or more closed curves. For point $\mathrm{P} \in M(A)$, we redefine $\beta(\mathrm{P})$ in the following way.

First, suppose that P has feet Q and Q$'$ that are on different boundary curves. In this case at least one of Q and Q$'$ is on the boundary of a hole. Then, we define $\beta(\mathrm{P}) = \infty$. This is because P is a midpoint of two different boundary curves and hence is necessary to keep the topology of the original figure. So we regard such P the most important, and will never prune off for any value of the threshold.

Next, suppose that both the feet of P belong to the same boundary curve. Then, we define the *support* of P and its *length* $l(\mathrm{P})$ in the same way as before. However, the normalized boundary distance defined for a simply connected figure is not suitable in this case, because boundary curves may have different lengths and consequently the normalization may give unnecessarily large value to P if the feet belong to a short boundary curve. In order to avoid this problem, we change the definition. Let $A_i, i = 1, 2, \ldots, k$, be a connected component of A, and L_i be the total length of the boundary ∂A_i. We define the normalized boundary distance as

$$\beta(\mathrm{P}) = \frac{l(\mathrm{P})}{\max\limits_{1 \le i \le k} L_i}. \tag{10}$$

This is a heuristic definition, and might be replaced with other definitions according to applications.

Fig. 5 shows an example of the behavior of the global medial axis for a general figure. This figure consists of two Chinese characters; the first one consists of a single connected component, and the second one consists of two connected components. All the three components have holes. In this figure, (a) shows the Voronoi diagram constructed in Step 2 of Algorithm 1, (b) shows the medial axis extracted in Step 3, and (c), (d), (e) and (f) represent the global medial axes for the threshold value $T = 0.03, 0.06, 0.12, 0.24$, respectively.

We can see that for any value of the threshold, the resulting structure is topologically equivalent to the original figure, and the larger T is, the more global part of the medial axis is extracted.

7. Concluding Remarks

We have proposed a new method for extracting the global medial axis in a stable manner. This algorithm is based on a new index for measuring importance, called the normalized boundary distance, which guarantees that the resulting global medial axis represents the same topological structure as the original figure.

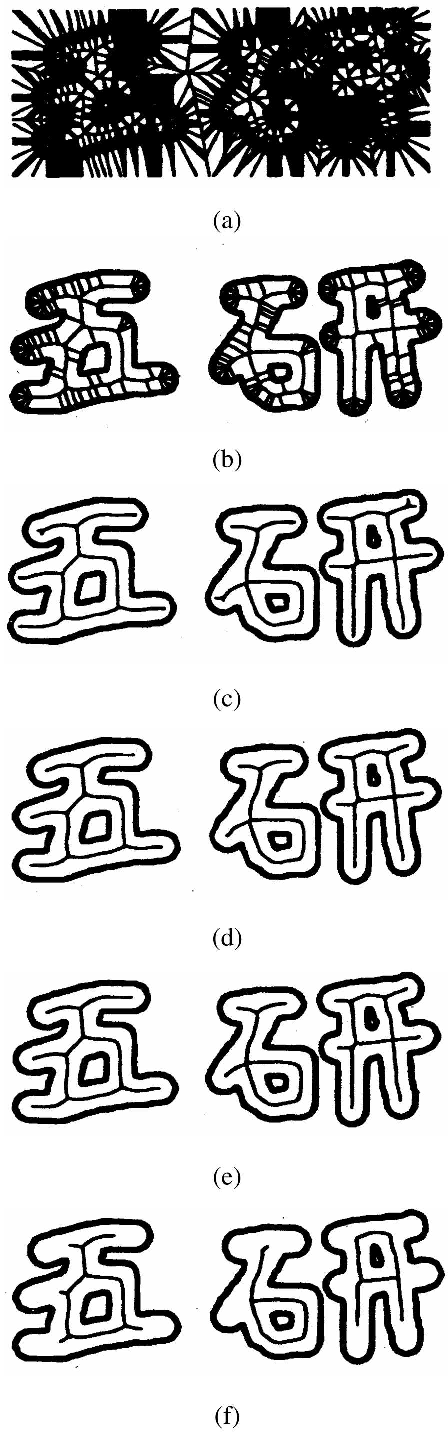


Figure 5. Example 2 (figure with more than one connected segments and holes): (a) Voronoi diagram; (b) Voronoi edges inside the boundary (primary medial axis); (c) edges with the index greater than 0.03; (d) edges with the index greater than 0.06; (e) edges with the index greater than 0.12; (f) edges with the index greater than 0.24.

Previous methods either require high cost in computation for smoothing the original figure in many different scales in order to get a hierarchical structure, or require heuristic post-processing in order to keep the same topology as the original figure. The proposed method overcomes both of the difficulties.

Actually, our new index, the normalized boundary distance, has a monotonicity property and hence the hierarchical structure can be obtained just by pruning with various threshold values, and moreover, the resultant structures guarantee the same topological structure as the original figure.

Problems for future contain investigation of other heuristic definitions of the degree of importance for a general figure, automatic selection of the threshold value, and applications of our method to practical problems.

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