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Optimal Lagrange interpolation by quartic C^1 splines on triangulations

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Abstract

We develop a local Lagrange interpolation scheme for quartic C^1 splines on triangulations. Given an arbitrary triangulation Δ, we decompose Δ into pairs of neighboring triangles and add "diagonals" to some of these pairs. Only in exceptional cases, a few triangles are split. Based on this simple refinement of Δ, we describe an algorithm for constructing Lagrange interpolation points such that the interpolation method is local, stable and has optimal approximation order. The complexity for computing the interpolating splines is linear in the number of triangles. For the local Lagrange interpolation methods known in the literature, about half of the triangles have to be split.

MSC: 41A05; 41A15; 65D05; 65D07; 65D17; 41A63

Keywords: Bivariate splines; Local Lagrange interpolation; Refinement of triangulations; Optimal approximation order

1. Introduction

There exists a vast literature on local Hermite interpolation, quasi-interpolation and related methods for splines on triangulations (cf. the survey of Nürnberger and Zeilfelder [24] and the references therein). On the other hand, local Lagrange interpolation methods for bivariate splines were developed only in the past five years [14,18–21,25–28,30]. The difficult problem of local Lagrange interpolation with bivariate splines was first formulated and discussed in the book of Chui [4]. This is an important problem in many fields of applications, since for the reconstruction of surfaces only data are used and no derivatives. In this connection, we note that it is known that if one would use a Hermite interpolation method (instead a Lagrange interpolation method), one would lose one approximation order. In particular, Hermite interpolation methods do not guarantee to obtain the approximate derivatives by some local method in the desired approximation order.

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We denote by $\mathscr{S}_q^r(\Delta)$ the space of splines of degree q and smoothness r on a triangulation Δ. The Lagrange interpolation problem for $\mathscr{S}_q^r(\Delta)$ is to construct sets $\{z_1, \ldots, z_d\}$ such that for any given data $f_1, \ldots, f_d$ a unique spline exists such that

$$s(z_i) = f_i, \quad i = 1, \ldots, d,$$

where d is the dimension of $\mathscr{S}_q^r(\Delta)$. Roughly speaking, local Lagrange interpolation means that for $i = 1, \ldots, d$, the change of the value f_i changes the interpolating spline s locally in a neighborhood of z_i.

A fundamental problem is to develop local Lagrange interpolation methods which have optimal approximation order. It is well known that for arbitrary triangulations Δ, the space $\mathscr{S}_q^r(\Delta)$ has optimal approximation order if $q \geqslant 3r + 2$ (cf. [7,16,10]). For $q < 3r + 2$, the approximation order of $\mathscr{S}_q^r(\Delta)$ for arbitrary triangulations Δ is not optimal (cf. [3]). In order to obtain optimal approximation order for the space $\mathscr{S}_4^1(\Delta)$, the triangulation Δ has to be modified (cf. [5,6], see also [11]). Non-local methods are developed in [1,9,23].

Local Lagrange interpolation methods with optimal approximation order were developed recently in [14,28,18,19,15] for the spaces $\mathscr{S}_q^1(\Delta)$, $q \geqslant 3$, and $\mathscr{S}_q^r(\Delta)$ for $r \geqslant 2$ and certain q, where about half of the triangles of Δ are split into three subtriangles. The resulting triangulation $\widehat{\Delta}$ has twice as much triangles as Δ.

In this paper, we describe the first local Lagrange interpolation method with optimal approximation order for $\mathscr{S}_4^1(\widetilde{\Delta})$, where only "diagonals" are added to Δ and in exceptional cases, a few triangles of Δ are split. In this case, the resulting triangulation $\widetilde{\Delta}$ has a simpler structure than $\widehat{\Delta}$. Moreover, the number of triangles in $\widetilde{\Delta}$ increases only by the factor of about $\frac{3}{2}$, since for about half of the quadrangles (formed of two triangles from Δ) a diagonal is added. The aim of the paper is to develop a local Lagrange interpolation method with optimal approximation order for quartic C^1 splines on arbitrary triangulations. Since this spline space does not possess optimal approximation order, the triangulation has to be refined.

The idea of our algorithm is as follows: given an arbitrary triangulation Δ, we decompose Δ into pairs of neighboring triangles i.e., convex and non-convex quadrangles, where only some isolated triangles remain. Then by considering common edges of these quadrangles, we create certain classes of quadrangles. Based on these classes, we add "diagonals" to some of the convex and non-convex quadrangles (see Fig. 7). Only in exceptional cases, a few triangles of Δ are split. The resulting triangulation is denoted by $\widetilde{\Delta}$. Then based on this decomposition, we choose interpolation points, first on the edges of the quadrangles and then in their interior.

In this way, we obtain a Lagrange interpolation set for $\mathscr{S}_4^1(\widetilde{\Delta})$. By using the above structure, we prove that the corresponding interpolation method is local, stable and has optimal approximation order, i.e.,

$$\|f - s\| \leqslant K \cdot h^5,$$

where h is the maximal diameter of the triangles in $\widetilde{\Delta}$, $\|.\|$ is a standard norm, and K is a constant depending on the smallest angle of Δ. In order to guarantee these properties, degenerate and near-degenerate edges have to be taken into special consideration. The complexity for computing the interpolating splines is linear in the number of triangles.

Local Lagrange methods are important for the construction and reconstruction of surfaces, since only data are needed and no derivatives. A standard approach is as follows. Given scattered data, in practice, one constructs a continuous linear spline based on a fine triangulation (which for example, represents a real world object) with approximation order h_{fine}^2 (where h_{fine} is the mesh size of the fine triangulation). This linear spline interpolates the given data. The corresponding fine triangulation depends on the data and is not regular, in general. Then by using the information of the linear splines, a coarse subtriangulation (which in general, is not a regular triangulation) with larger mesh size h is constructed by using mesh simplification methods (see [8,13], for instance). By applying Lagrange interpolation one constructs a quartic C^1 interpolating spline defined on the coarse subtriangulation which interpolates the linear spline. The data are taken directly from the linear spline. This is done such that, in our case, h_{fine}^2 is about h^5. In this way, significant data compression rates are obtained. Obviously, this method is more effective than to use a regular coarse triangulation and the C^1 spline interpolates at a subset of characteristic points.

Numerical tests of local Lagrange interpolation methods for bivariate splines (involving test functions as well as real world data) and further details on the implementation of our approach were given in Nürnberger et al. [21], Nürnberger and Zeilfelder [26,28], and Zeilfelder [31] (see also [23–25,30]). The tests showed that these interpolation methods lead to significant data compressions and work efficiently, i.e., the interpolating splines can be computed on a standard

PC for large data sets with up to millions of points. Moreover, in Nürnberger and Zeilfelder [28] examples comparing Lagrange and Hermite interpolation are discussed.

The paper is organized as follows. In Section 2, we describe quartic C^1 splines, its piecewise Bézier–Bernstein form, the interpolation problem, and smoothness conditions. Algorithms for decompositing and refining arbitrary triangulations are given in Section 3. Our approach of constructing local Lagrange interpolation points and the main result are presented in Section 4. In Section 5, we give some auxiliarly results, which we use in the proof of the main theorem to be found in the final section. We conclude the final section by giving an error bound of our interpolation method and a remark.

2. Spline spaces and interpolation

Throughout the paper, we consider the space of *bivariate C^1-splines of degree* 4 (quartic C^1 splines) with respect to Δ, defined as

$$\mathscr{S}_4^1(\Delta) = \{s \in C^1(\Omega): s_{|T} \in \mathscr{P}_4,\, T \in \Delta\},$$

where $\mathscr{P}_4 = \mathrm{span}\{x^i y^j : i,\, j \geqslant 0,\, i+j \leqslant 4\}$ denotes the space of *bivariate polynomials of degree* 4 and $C^1(\Omega)$ the space of all continuously differentiable functions on Ω.

We use the well-known *Bézier–Bernstein representation* of bivariate splines. Given a triangle $T = \Delta(v_1, v_2, v_3)$ in Δ, let

$$D_T := \{P_{i,j,k}^{[T]} := (iv_1 + jv_2 + kv_3)/4 : i + j + k = 4\}$$

be the set of *domain points* on T. Then, every quartic spline s can be written as

$$s_{|T}(z) = \sum_{i+j+k=4} a_{i,j,k}^{[T]} \cdot B_{i,j,k}(z), \quad z \in T, \tag{1}$$

where

$$B_{ijk}(z) = 4!/(i!j!k!)\phi_1^i(z)\phi_2^j(z)\phi_3^k(z)$$

are the *Bernstein* (*basis*) *polynomials* of degree 4 associated with T. Here, ϕ_m, $m = 1, 2, 3$ are the unique linear polynomials satisfying the interpolation property $\phi_m(v_l) = \delta_{ml}$, $l = 1, 2, 3$ called *barycentric coordinates*. It is well known that every continuous quartic spline is uniquely determined by the *Bézier Bernstein coefficients* $a_{i,j,k}^{[T]}$, $i+j+k=4$, $T \in \Delta$.

A set $\mathscr{L} = \{z_1, \ldots, z_d\} \subseteq \Omega$, where d is the dimension of a spline space $\mathscr{S}_4^1(\Delta)$, is called a *Lagrange interpolation set for* $\mathscr{S}_4^1(\Delta)$, if for any given data $f_1, \ldots, f_d$, a unique spline $s \in \mathscr{S}_4^1(\Delta)$ exists such that

$$s(z_i) = f_i, \quad i = 1, \ldots, d. \tag{2}$$

A Lagrange interpolation method for $\mathscr{S}_4^1(\Delta)$ is called *n-local* if there exists an integer n such that for each Bézier Bernstein coefficient $a_{i,j,k}$ of the interpolating spline $s \in \mathscr{S}_4^1(\Delta)$, the following condition holds: there exists a suitable vertex v such that $a_{i,j,k}$ is uniquely determined using only the interpolation values in $\mathscr{L}_{i,j,k} := \mathscr{L} \cap st^n(v)$. Here $st(v) = st^1(v) \subseteq \Omega$ denotes the union of all triangles with vertex v and

$$st^n(v),\, n \geqslant 2, \tag{3}$$

is defined inductively as the union of all triangles in Δ that intersect $st^{n-1}(v)$ (see Fig. 1). We say that a Lagrange interpolation method for $\mathscr{S}_4^1(\Delta)$ is *local*, if the method is n-local for a suitable n. Moreover, such an interpolation method is called *stable*, if there exists a constant C depending only on the smallest angle in Δ such that each Bézier Bernstein coefficient $a_{i,j,k}$ of the interpolating spline $s \in \mathscr{S}_4^1(\Delta)$, satisfies

$$|a_{i,j,k}| \leqslant C \cdot \max_{z \in \mathscr{L}_{i,j,k}} |f(z)|. \tag{4}$$

In the following, we recall a well-known result (cf. [2,4,12]) in our setting of quartic C^1 splines. The next theorem characterizes the C^1 smoothness of two bivariate quartic polynomial pieces in their Bézier–Bernstein representation on neighboring triangles across the common edge.

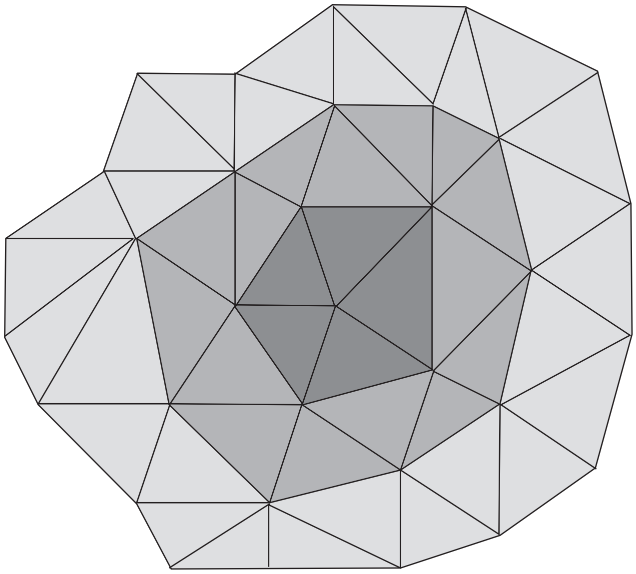

Fig. 1. Stars for a vertex v. The region $st(v)$, $st^2(v)$ and $st^3(v)$ is marked dark gray, bright gray and white, respectively.

Theorem 1. *Let two neighboring triangles $T_1 = \Delta(v_1, v_2, v_3)$ and $T_2 = \Delta(v_1, v_2, v_4)$ and a continuous spline s on $T_1 \cup T_2$ with*

$$s_{|T_1} = p_1 = \sum_{i+j+k=4} a_{i,j,k}^{[T_1]} B_{i,j,k} \quad \textit{and} \quad s_{|T_2} = p_2 = \sum_{i+j+k=4} a_{i,j,k}^{[T_2]} B_{i,j,k},$$

be given. Then, s is differentiable across the common edge $[v_1, v_2]$, if and only if for all $i + j = 3$,

$$a_{i,j,1}^{[T_2]} = a_{i+1,j,0}^{[T_1]}\phi_1(v_4) + a_{i,j+1,0}^{[T_1]}\phi_2(v_4) + a_{i,j,1}^{[T_1]}\phi_3(v_4). \tag{5}$$

We note that it follows immediately from (5) that

$$\begin{aligned}|a_{i,j,1}^{[T_2]}| &= \left|a_{i+1,j,0}^{[T_1]}\phi_1^{[T_1]}(v_4) + a_{i,j+1,0}^{[T_1]}\phi_2^{[T_1]}(v_4) + a_{i,j,1}^{[T_1]}\phi_3^{[T_1]}(v_4)\right| \\ &\leqslant C \cdot \max\{|a_{i+1,j,0}^{[T_1]}|, |a_{i,j+1,0}^{[T_1]}|, |a_{i,j,1}^{[T_1]}|\},\end{aligned} \tag{6}$$

and, if $\phi_1^{[T_1]}(v_4) \neq 0$,

$$\begin{aligned}|a_{i+1,j,0}^{[T_1]}| &= \left|(1/\phi_1^{[T_1]}(v_4))(a_{i,j,1}^{[T_2]} - a_{i,j+1,0}^{[T_1]}\phi_2^{[T_1]}(v_4) - a_{i,j,1}^{[T_1]}\phi_3^{[T_1]}(v_4))\right| \\ &\leqslant \widetilde{C} \cdot \max\{|a_{i,j,1}^{[T_2]}|, |a_{i,j+1,0}^{[T_1]}|, |a_{i,j,1}^{[T_1]}|\},\end{aligned} \tag{7}$$

where $i + j = 3$. The constant C depends only on the smallest angle in T_1. Moreover, it is obvious that the constant $\widetilde{C}$ depending on the angle $\angle(v_3, v_1, v_4)$ is non-zero, if it exists, i.e., $\phi_1^{[T_1]}(v_4) \neq 0$, or equivalently, the vertices v_2, v_3, and v_4 do not lie on a common line.

In addition, we use the following important notions throughout the paper. Two triangles T_1, T_2 from Δ are called *neighboring triangles* if they share a common edge e. An edge e is called *degenerate* at a vertex v if the remaining edges of T_1 and T_2 with endpoint v have the same slope. Moreover, choosing a small $\varepsilon > 0$, we call an edge *e near-degenerate* at the vertex v, if the angle between the remaining edges of T_1 and T_2 with endpoint v is in $[\pi - \varepsilon, \pi + \varepsilon]$. We choose ε as

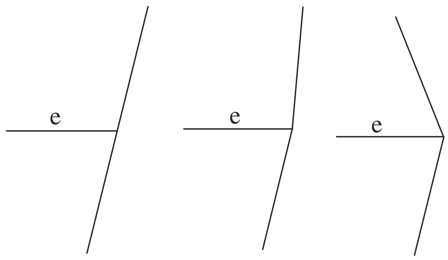

Fig. 2. Degenerate, near-degenerate and proper edge (from left to right).

small as possible but big enough such that the Bézier–Bernstein coefficients of the interpolating splines can be computed in a stable way. If e is neither degenerate nor near-degenerate at both vertices, then we call e a *proper* edge. Fig. 2 illustrates these notions. The algorithms described in the next section (decomposition of triangulation, classification of quadrangles) result in a refinement $\tilde{\Delta}$ of the given triangulation Δ, where we add some additional degenerate edges and take care of the near-degenerate case.

3. Decomposition and refinement of triangulations

In Section 4, we describe our algorithm for constructing local interpolation sets for quartic C^1 splines. For doing this, we first decompose a given triangulation by considering degenerate and near-degenerate edges. Based on this decomposition, we create classes of quadrangles and then refine the triangulation.

3.1. Decomposition of triangulations

Let a *triangulation* Δ of a polygonal domain $\Omega \subseteq \mathbb{R}^2$ be given i.e., Δ is a set of closed triangles such that each non-empty intersection of two triangles in Δ is a common edge or a common vertex.

In the following, we decompose Δ into three disjoint sets denoted by Δ_T, Δ_Q^1, and Δ_Q^2. Δ_Q^1 and Δ_Q^2 consist of quadrangles Q formed by pairs of neighboring triangles T_1, T_2 in Δ. Δ_Q^2 contains quadrangles with proper diagonals, while the quadrangles in Δ_Q^1 contain degenerate or near-degenerate diagonals. Our algorithm guarantees that any two quadrangles from Δ_Q^1 do not have a common edge. Δ_T consists of some remaining *isolated triangles* not contained in Δ_Q^1 and Δ_Q^2.

Algorithm 1 (*Decomposition of Δ*). (i) Let $e_1, \ldots, e_n$ be the interior edges of the triangulation Δ which are proper and let the sets Δ_Q^1, Δ_Q^2 and Δ_T be empty. Let all triangles of Δ be unmarked. (ii) Start an inductive procedure with e_1 by putting the quadrangle $Q = T_{1,1} \cup T_{1,2}$ into Δ_Q^2, where $T_{1,1}$ and $T_{1,2}$ are the neighboring triangles of Δ which have e_1 as a common edge. Mark the triangles $T_{1,1}$ and $T_{1,2}$. (iii) For $i = 2, \ldots, n$, apply the inductive step with e_i as follows. If $T_{i,1}$ and $T_{i,2}$ are the neighboring triangles of Δ, which have e_i as a common edge *and* both triangles are unmarked, then put the quadrangle $Q = T_{i,1} \cup T_{i,2}$ into Δ_Q^2, and mark the triangles $T_{i,1}$ and $T_{i,2}$. Otherwise, omit e_i. (iv) Put all unmarked triangles of Δ into Δ_T which have no unmarked neighbors. Mark these triangles. (v) If $T_{i,1}$ and $T_{i,2}$ are two unmarked neighboring triangles of Δ such that all other neighboring triangles of Δ are marked, then put $Q = T_{i,1} \cup T_{i,2}$ into Δ_Q^1. Mark the triangles $T_{i,1}$ and $T_{i,2}$. (vi) The remaining unmarked triangles of Δ form chains of length at least three. Add edges as shown in Fig. 3 and put quadrangles and triangles into Δ_Q^2, Δ_Q^1, and Δ_T according to steps (iii), (iv), and (v) of the algorithm. (vii) If no unmarked triangles of Δ remain, the algorithm stops.

We illustrate Algorithm 1 with two examples shown in Figs. 4 and 5. In these figures the quadrangles from Δ_Q^2 are marked gray, while the quadrangles with (near-)degenerate diagonals i.e., quadrangles in Δ_Q^1, are marked white, and Δ_T is illustrated by black triangles. We note that the quadrangles in Δ_Q^1 are identified by the algorithm, since we have to be free to choose appropriate interpolation points (see next section) close to the near-degenerate edges. These

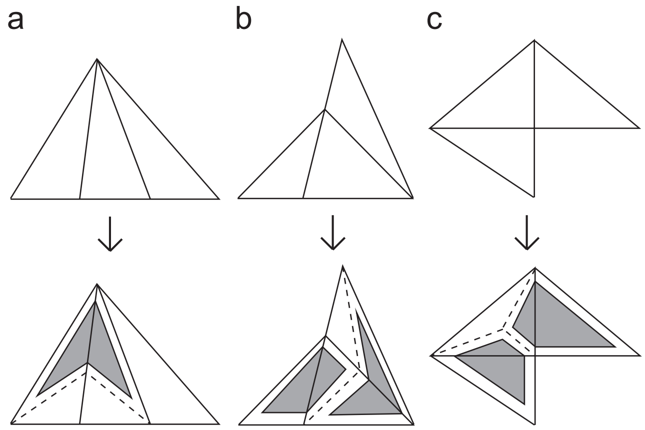

Fig. 3. The remaining unmarked triangles after step (v) of Algorithm 1 yield chains of length at least three. In this case, the interior edges are degenerate or near-degenerate at some vertex such that no pairs of triangles for Δ_Q^2 can be combined. Considering three such triangles, three cases can occur. In the cases (a) and (b) the second "diagonale" and in the case (c) a Clough–Tocher split creates further proper edges and quadrangles that break the chains.

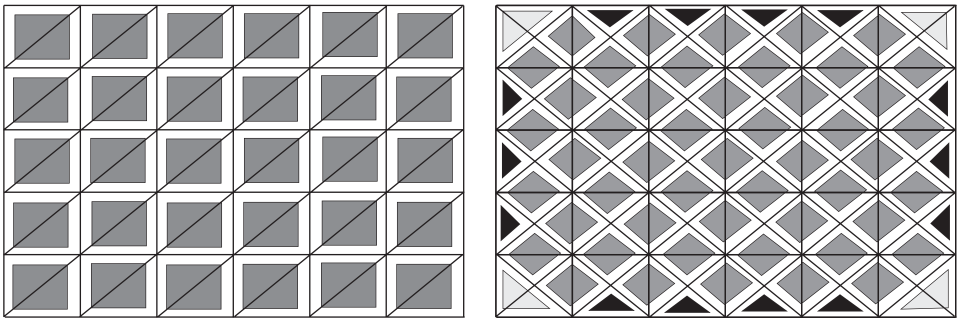

Fig. 4. Decomposition of uniform triangulations, i.e., three- and four-directional meshes.

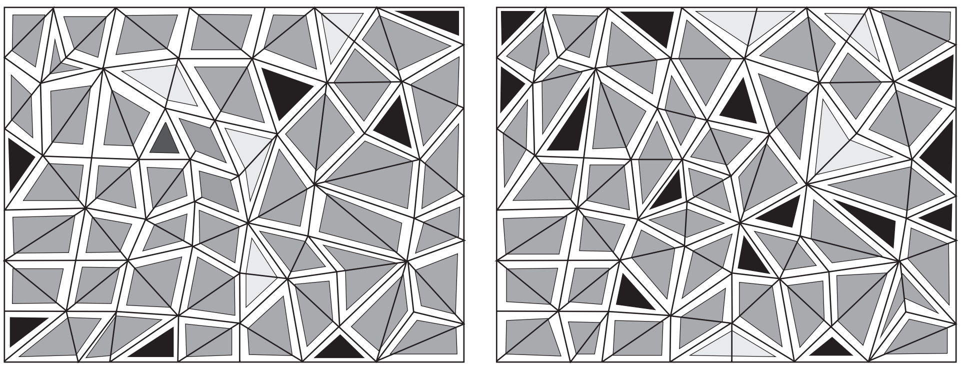

Fig. 5. Decompositions of a general triangulation. The quadrangles from Δ_Q^2 are colored gray, while the quadrangles with (near-) degenerate diagonals i.e., quadrangles in Δ_Q^1, are white. Triangles in Δ_T are marked black. The decomposition obtained from Algorithm 1 is not unique in general.

examples show that in general, only steps (i)–(v) are necessary for the complete decomposition of the triangulation. However, there exist exceptional cases with many degenerate edges, such that step (vi) has to be applied. We remark that, if (vi) has to be applied, it immediately follows from steps (iv) and (v) that the remaining unmarked triangles form

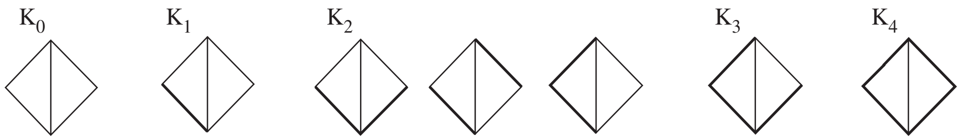

Fig. 6. From the left to the right: quadrangles in the class $\mathscr{K}_j$, $j=0,\dots,4$. $\mathscr{K}_2$ consists of three subclasses. Common edges with quadrangles in $\mathscr{K}_0,\dots,\mathscr{K}_{j-1}$ and triangles of Δ_T, respectively, are illustrated by thicker lines.

chains of length at least three, which are treated inductively. In particular, Fig. 3 shows which edges are added to Δ, when this case occurs (see also Section 3.3). Also note that it is clear that Δ_T consists of isolated triangles, because otherwise a further quadrangle would have been chosen by the algorithm. Moreover, it is obvious that the decomposition obtained by applying Algorithm 1 is not unique in general (see Fig. 5). On the other hand, each decomposition has more or less the same properties, namely the number of quadrangles in Δ_Q^1 is essentially smaller than the number of quadrangles in Δ_Q^2, and Δ_T consists of a relatively small set of triangles. The four-directional mesh (Fig. 4, right) is a quite typical example with many degenerate edges, but the algorithm creates many quadrangles in Δ_Q^2, because proper edges are considered first.

3.2. Classification according to common edges

Based on the decomposition of Δ from the previous subsection, we further classify the quadrangles of Δ_Q^1 and Δ_Q^2. Roughly speaking, we construct five classes $\mathscr{K}_j$, $j=0,\dots,4$, of quadrangles such that each $Q \in \mathscr{K}_j$ has exactly j common edges with quadrangles in $\mathscr{K}_l$, $l=0,\dots,j$. Here, we use the *priority principles* introduced in [28] (see also [14,18,19,31]) which is the key for the locality of our interpolation method (Fig. 6).

Algorithm 2 (*Classification of quadrangles*). Step (i) Start with $\mathscr{K}_j=\emptyset$, $j=0,\dots,4$. Put all quadrangles of Δ_Q^1 into $\mathscr{K}_0$ and mark all triangles in Δ_T and all quadrangles in Δ_Q^1. Let the quadrangles $Q_1^1,\dots,Q_{m_1}^1$ of Δ_Q^2 be unmarked. For $j=1,\dots,4$ proceed with step $(j+1)$, which is as follows. Let $Q_1^j,\dots,Q_{m_j}^j$ be the remaining unmarked quadrangles. For $i=1,\dots,m_j$ consider Q_i^j. If Q_i^j has exactly $j-1$ common edges with the marked triangles and quadrangles, then put Q_i^j into the class $\mathscr{K}_{j-1}$ and mark Q_i^j. Otherwise, omit Q_i^j. Step (vi) Put the remaining unmarked quadrangles into $\mathscr{K}_4$.

The different classes of quadrangles produced by Algorithm 2 are shown in Fig. 6. Note that quadrangles containing near-degenerate edges are put into class $\mathscr{K}_0$. The reason for doing this is that these edges have to be handled with care to guarantee the stability of the interpolation method, and our approach described below shows that this can be done by choosing an appropriate set of interpolation points in the quadrangles of Δ_Q^1.

Lemma 2. *The following statements hold*:

(i) *No two elements in $\Delta_T \cup \mathscr{K}_0$ have a common edge.*
(ii) *All quadrangles in Δ_Q^1 are contained in $\mathscr{K}_0$.*
(iii) *No two quadrangles in $\mathscr{K}_j$, $j \in \{1,\dots,4\}$, have a common edge.*

Proof. According to steps (iv) and (v) of Algorithm 1, no two elements in $\Delta_T \cup \Delta_Q^1$ have a common edge. Moreover, by step (ii) of Algorithm 2 a quadrangle in Δ_Q^2 has no marked neighbor, when it is put into $\mathscr{K}_0$. This proves (i). Statement (ii) is an immediate consequence of step (i) of Algorithm 2. Now, let us assume that two quadrangles Q_1 and Q_2 in $\mathscr{K}_j$, $j \in \{1,\dots,4\}$, have a common edge. Then according to Algorithm 2, both quadrangles had exactly j marked neighbors when they are chosen by the algorithm. We may assume that Q_1 has been considered before Q_2. Hence, Q_1 was already marked, when considering Q_2. This implies that before Q_1 is considered, Q_2 has only $j-1$ marked neighbors, such that Q_2 would have been put into $\mathscr{K}_{j-1}$ in an earlier step. This is a contradiction. This completes the proof. $\square$

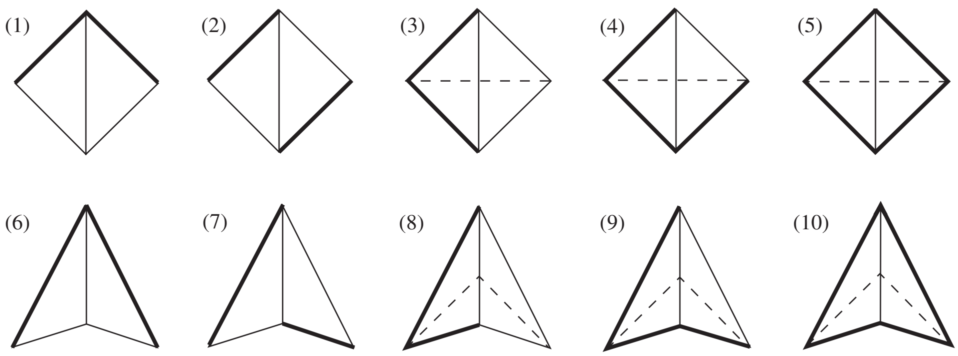

Fig. 7. Refinement of the quadrangles in $\mathscr{K}_j$, $j = 2, \ldots, 4$. The edges in common with neighboring quadrangles from $\mathscr{K}_i$, $i = 0, \ldots, j-1$, and triangles from Δ_T are shown as thicker lines. The edges added to define $\tilde{\Delta}$ are shown as dashed lines. (1), (2), (6), and (7) show the convex and non-convex cases, where a quadrangle in $\mathscr{K}_2$ is not refined. In these cases the thick edges belong to different triangles. In the convex cases (3), (4), and (5), the second diagonal is added. In the non-convex cases (8), (9), and (10) the midpoint of the diagonal is connected to the remaining two vertices.

3.3. Refinement of triangulation

In this subsection, we describe a refinement of the triangulation suitable for our local Lagrange interpolation method. Let Δ be an arbitrary triangulation and the sets Δ_Q^1, Δ_Q^2, and Δ_T as well as the classes $\mathscr{K}_j$, $j = 0, \ldots, 4$, be constructed as in the previous subsections. We emphasize that it follows from the above that in some exceptional cases a few triangles of Δ have already been refined.

Consider each quadrangle in Δ_Q^2. In two cases Q will be refined. (i) Q is in $\mathscr{K}_3$ or $\mathscr{K}_4$, or (ii) Q is in $\mathscr{K}_2$ and moreover, both edges of Q in common with some quadrangles and triangles in $\mathscr{K}_0$, $\mathscr{K}_1$, and Δ_T, respectively, are edges of the same triangle in contained in Q (see Fig. 7, (3) and (8)).

Let Q be such a quadrangle and $[u, v]$ be its diagonal. We refine Q as follows. (i) If Q is convex, then we add the second diagonal i.e., we connect the remaining two vertices of Q. (ii) If Q is non-convex, then we connect the remaining two vertices of Q with the midpoint $(\frac{1}{2})(u + v)$ of its diagonal. We denote the refined triangulation by $\tilde{\Delta}$.

Remark. If we add the second diagonal to a convex quadrangle Q, the angles of the resulting refined partition are only bounded from below by $\varepsilon/2$. Hence, in particular cases where angles $\leqslant \varepsilon$ may appear, we treat such quadrangles in the same way as described in (ii) for the non-convex case. For simplicity, we did not formulate this as a special rule in the above refinement.

4. Construction of interpolation points

We choose points for unique interpolation by quartic C^1 splines on $\tilde{\Delta}$. First, we choose points on the edges of the refined triangulation $\tilde{\Delta}$. These points uniquely determine the interpolating spline on all edges of the quadrangles in Δ_Q^1, but only on the boundary edges of the quadrangles in Δ_Q^2. In a second step, interpolation points are chosen in the interior of the quadrangles based on the corresponding class.

Let $\tilde{\Delta}$, Δ_Q^1, and Δ_Q^2 be constructed as described in Section 3. First, we create an additional classification $\mathscr{N}_j$, $j = 0, \ldots, 4$, of the quadrangles in Δ_Q^1 and Δ_Q^2, with respect to common vertices. Then according to these classes, we choose the interpolation points on the edges.

Algorithm 3 (*Classification with respect to common vertices*). (i) Start with $\mathscr{N}_j = \emptyset$, $j = 0, \ldots, 4$, and let all vertices of $\tilde{\Delta}$ be unmarked. (ii) For $j = 0, \ldots, 3$ consider the quadrangles $Q_1, \ldots, Q_n$ of Δ_Q^1 and Δ_Q^2. If Q_i has exactly j marked vertices, then put Q_i into the class $\mathscr{N}_j$ and mark the vertices of Q_i. Otherwise, omit Q_i. (iii) The remaining quadrangles yield the class $\mathscr{N}_4$.

The following observations are essential for the locality of our interpolation method.

Lemma 3. *The following statements hold*:

(i) *No two quadrangles in* $\mathcal{N}_0$ *have a common vertex.*
(ii) *If two quadrangles in* $\mathcal{N}_j$, $j = 1, \ldots, 4$, *have a common vertex* v, *then there exists a quadrangle in* $\mathcal{N}_i$, $i < j$, *with vertex* v.

Proof. Since all vertices of a quadrangle assigned to $\mathcal{N}_0$ have to be unmarked, no vertex of Δ can be a vertex of two (different) quadrangles in $\mathcal{N}_0$. This proves (i). Now, let Q_1, Q_2 be two quadrangles in $\mathcal{N}_j$, $j \in \{1, \ldots, 4\}$, with a common vertex v. We may assume that Q_1 has been assigned before Q_2. Then, v is marked when Q_2 is assigned by Algorithm 3. Now, let us assume that v was also unmarked before Q_1 has been assigned. Then before Q_1 was assigned, Q_2 would have had v and $j - 1$ unmarked vertices differing from v. In this case Q_2 would have been put into $\mathcal{N}_{j-1}$, in an earlier step. This is a contradiction. Therefore, v was already marked when Q_1 has been considered. Hence, there exists a quadrangle in $\mathcal{N}_i$, $i < j$, with vertex v. This completes the proof. □

In the following, we construct Lagrange interpolation points for $\mathcal{S}_4^1(\widetilde{\Delta})$. We first choose points on the edges and then in the interior of the triangles of $\widetilde{\Delta}$.

Let Q be a quadrangle consisting of two triangles $T_1 = \Delta(v_1, v_2, v_3)$ and $T_2 = \Delta(v_1, v_3, v_4)$. We may assume that $v_1, \ldots, v_k$, $k \in \{1, \ldots, 4\}$, are the marked vertices of Q, as Q was assigned to $\mathcal{N}_k$. Then we define $\mathcal{L}_Q^e$ as the set of points $((4 - \alpha_j)v_j + \alpha_j v_{j+1})/4$, $v_5 := v_1$, where (i) $\alpha_j = 0, \ldots, 3$, $j = 1, \ldots, 4$, if Q is in $\mathcal{N}_0$, (ii) $\alpha_1 = 2, 3$, $\alpha_2, \alpha_3 = 0, \ldots, 3$, $\alpha_4 = 0, 1, 2$, if Q is in $\mathcal{N}_1$, (iii) $\alpha_1 = 2$, $\alpha_2 = 2, 3$, $\alpha_3 = 0, \ldots, 3$, $\alpha_4 = 0, 1, 2$, if Q is in $\mathcal{N}_2$, (iv) $\alpha_1, \alpha_2 = 2$, $\alpha_3 = 2, 3$, $\alpha_4 = 0, 1, 2$, if Q is in $\mathcal{N}_3$, and (v) $\alpha_j = 2$, $j = 1, \ldots, 4$, if Q is in $\mathcal{N}_4$ (see Fig. 8, upper row).

Moreover, we replace $(3v_1 + v_2)/4$ (if contained in $\mathcal{L}_Q^e$) by $(3v_1 + v_3)/4$, and replace $(v_2 + 3v_3)/4$ (if contained in $\mathcal{L}_Q^e$) by $(v_1 + 3v_3)/4$, and add $(v_1 + v_3)/2$, if the quadrangle Q is in Δ_Q^1 i.e., if its interior edge is degenerate or near-degenerate (see Fig. 8, lower row).

Moreover, for each triangle $T = \Delta(v_1, v_2, v_3)$ in Δ_T, we define $\mathcal{L}_T^e$ to be the set of points v_1 and $((4 - \alpha)v_1 + \alpha v_j)/4$, $\alpha = 1, 2$, $j = 2, 3$, if T has two boundary edges $[v_1, v_2]$ and $[v_1, v_3]$, and the point $(v_1 + v_2)/2$, if T has exactly one boundary edge $[v_1, v_2]$. Otherwise, let $\mathcal{L}_T^e$ be empty.

Finally, let

$$\mathcal{L}^e := \left(\bigcup_{Q \in \Delta_Q^1 \cup \Delta_Q^2} \mathcal{L}_Q^e \right) \cup \left(\bigcup_{T \in \Delta_T} \mathcal{L}_T^e \right). \tag{8}$$

Now, we choose additional points in the interior of the quadrangles. Let Q be a quadrangle consisting of two triangles $T_1 = \Delta(v_1, v_2, v_3)$ and $T_2 = \Delta(v_1, v_3, v_4)$. If Q is refined (see Section 3.3), then we denote the triangles of Q by

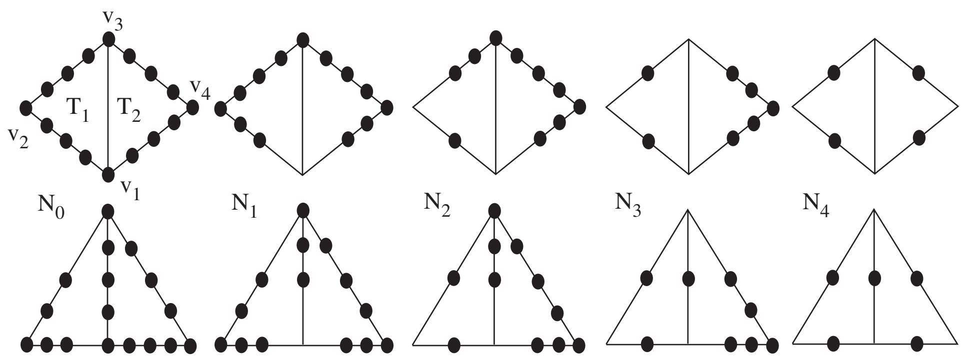

Fig. 8. Chosen points on the edges of the quadrangles. Each N_j, $j = 0, \ldots, 4$, represents a quadrangle in the class $\mathcal{N}_j$. The points on the edges of the quadrangles in Δ_Q^2 (upper row) and the quadrangles in Δ_Q^1 (lower row) are marked by black dots. From the left to the right: none, one, two, and three vertices of the quadrangles are marked by Algorithm 3. For illustrating purposes we assume here that $v_1, \ldots, v_j$ are the marked vertices, if the quadrangle is in $\mathcal{N}_j$.

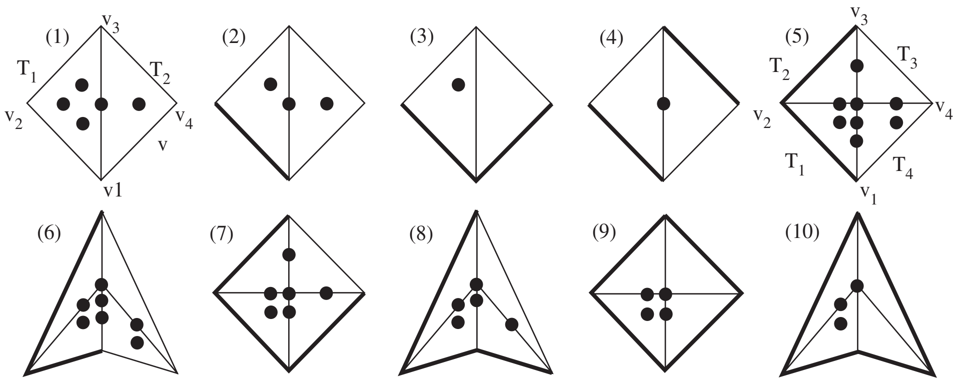

Fig. 9. Lagrange points chosen in the interior of the quadrangles are marked by black dots. Marked edges of the quadrangles are illustrated by thicker lines. In (1)–(4), the quadrangles are not refined and the choice of interpolation points coincides in the convex and non-convex case. In (5)–(10), where the quadrangles are refined, one additional interpolation point is chosen in the convex case.

$T_j = \Delta(z, v_j, v_{j+1})$, $j = 1, \ldots, 4$, where $v_5 := v_1$, and z is the new vertex in the interior of Q. We define $\mathscr{L}^i_Q$ as a set of points in Q in correspondence to the assigned class $\mathscr{K}_j$, $j \in \{0, \ldots, 4\}$ of Algorithm 2. Seven cases occur:

(i) $Q \in \mathscr{K}_0$. Then, we choose the five points $(2v_1 + v_2 + v_3)/4$, $(v_1 + 2v_2 + v_3)/4$, $(v_1 + v_2 + 2v_3)/4$, $(v_1 + v_3)/2$, and $(v_1 + v_3 + 2v_4)/4$ (see Fig. 9, (1)). (ii) $Q \in \mathscr{K}_1$. Let $[v_1, v_2]$ be the marked edge when assigning Q. Then, we choose the three points $(v_1 + v_2 + 2v_3)/4$, $(v_1 + v_3)/2$, and $(v_1 + v_3 + 2v_4)/4$ (see Fig. 9, (2)). (iii) $Q \in \mathscr{K}_2$. Let $[v_1, v_2]$ and $[v_1, v_4]$ be the marked edge when Q was assigned. Then, we choose the point $(v_1 + v_2 + 2v_3)/4$ (see Fig. 9, (3)). (iv) $Q \in \mathscr{K}_2$. Let $[v_1, v_2]$ and $[v_3, v_4]$ be the marked edge when Q was assigned. Then, we choose the point $(v_1 + v_3)/2$ (see Fig. 9, (4)). In the remaining cases Q is refined. (v) $Q \in \mathscr{K}_2$. Let $[v_1, v_2]$ and $[v_2, v_3]$ be the marked edge when Q was assigned. Then, we choose the points $((3-\alpha)v_1 + v_2 + \alpha z)/4$, $\alpha = 2, 3$, $((4-\alpha)v_1 + \alpha z)/4$, $\alpha = 2, 3, 4$, and $((2-\alpha)v_1 + \alpha z + 2v_4)/4$, $\alpha = 1, 2$. Moreover, we choose $(z + v_3)/2$ if and only if Q is convex (see Fig. 9, (5) and (6)). (vi) $Q \in \mathscr{K}_3$. Let $[v_1, v_2]$, $[v_2, v_3]$, and $[v_4, v_1]$ be the marked edge when Q was assigned. Then, we choose the points $((3-\alpha)v_1 + v_2 + \alpha z)/4$, $\alpha = 2, 3$, $((4-\alpha)v_1 + \alpha z)/4$, $\alpha = 3, 4$, and $(z + v_4)/2$. Moreover, we choose $(z + v_3)/2$ if and only if Q is convex (see Fig. 9, (7) and (8)). (vii) $Q \in \mathscr{K}_4$. Then, we choose the points $((3-\alpha)v_1 + v_2 + \alpha z)/4$, $\alpha = 2, 3$, and z. Moreover, we choose $(v_1 + 3z)/4$ if and only if Q is convex (see Fig. 9, (9) and (10)).

Moreover, for each triangle $T = \Delta(v_1, v_2, v_3)$ in Δ_T let $\mathscr{L}^i_T$ be the three points $(2v_1 + v_2 + v_3)/4$, $(v_1 + 2v_2 + v_3)/4$, and $(v_1 + v_2 + 2v_3)/4$.

This leads to the following set of points chosen in the interior of the quadrangles and triangles (Fig. 10):

$$\mathscr{L}^i := \left(\bigcup_{Q \in \Delta^1_Q \cup \Delta^2_Q} \mathscr{L}^i_Q \right) \cup \left(\bigcup_{T \in \Delta_T} \mathscr{L}^i_T \right). \tag{9}$$

Now, we are ready to state the main result of our paper. The proof will be given in Section 6. Let $\widetilde{\Delta}$ be the refinement of an arbitrary triangulation Δ, described in Section 3.3. Moreover, let the sets $\mathscr{L}^e$ and $\mathscr{L}^i$ be as defined in (8) and (9), respectively.

Theorem 4. $\mathscr{L} := \mathscr{L}^e \cup \mathscr{L}^i$ *is a Lagrange interpolation set for* $\mathscr{S}^1_4(\widetilde{\Delta})$. *Moreover, the interpolation method is local and stable.*

An example for an interpolation set is shown in Fig. 10.

5. Auxiliarly results

In this section we establish some auxiliarly results which are essential for the proof of the main theorem (Theorem 4) given in the next section. Here, we consider the computation of the Bézier–Bernstein coefficients corresponding to

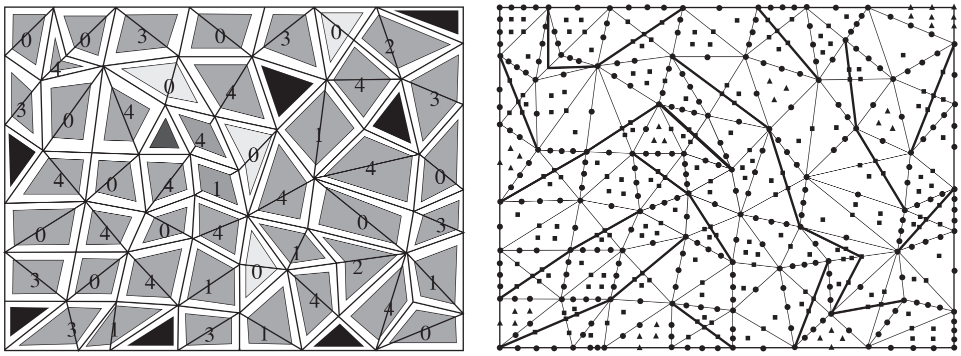

Fig. 10. Classes of quadrangles for the triangulation in Fig. 5. The quadrangles in $\mathcal{K}_j$, $j=0,\dots,4$, are indicated by the corresponding numbers (left). An example of an interpolation set for $\mathcal{S}_4^1(\widetilde{\Delta})$ resulting from our method is shown on the right. Interpolation points chosen on the edges of the quadrangles, in the interior of the quadrangles, and in the interior of the triangles in Δ_T are indicated by black dots, black squares, and black triangles, respectively.

interior domain points of the quadrangles, where the five classes $\mathcal{K}_j$, $j=0,\dots,4$, resulting from Algorithm 2 are taken into consideration. In what follows, we let $\Delta(Q)=\{T_1,T_2\}$, where $T_1=\Delta(v_1,v_2,v_3)$ and $T_2=\Delta(v_1,v_3,v_4)$.

Lemma 5 ($Q\in\mathcal{K}_0$, *see Fig. 9, (1)*). *Let all coefficients* $a_{i,j,k}^{[T_m]}$, $m=1,2$, *with* $i=0$, $j=0$, *or* $k=0$ *of a spline* $s\in\mathcal{S}_4^1(\Delta(Q))$ *be determined. Then, the interpolation values* $f(P)$, $P\in\mathcal{L}_Q^i\backslash\{(v_1+v_3)/2\}$, *uniquely determine the six Bézier–Bernstein coefficients* $a_{i,j,k}^{[T_m]}$, $i,j,k\geqslant 1$, $m=1,2$.

Proof. The coefficients $a_{1,1,2}^{[T_1]}, a_{1,2,1}^{[T_1]}$, and $a_{2,1,1}^{[T_1]}$, are computed from the values $f(P)$ at the points $P\in\{(2v_1+v_2+v_3)/4, (v_1+2v_2+v_3)/4, (v_1+v_2+2v_3)/4\}$ by solving a system of three linear equations. This system can be written as

$$\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\cdot x=y, \tag{10}$$

where $x=(a_{1,1,2}^{[T_1]}, a_{1,2,1}^{[T_1]}, a_{2,1,1}^{[T_1]})^t$ and $y\in\mathbb{R}^3$ is uniquely determined by the values $f(P)$, $P\in\{(2v_1+v_2+v_3)/4, (v_1+2v_2+v_3)/4, (v_1+v_2+2v_3)/4\}$, and the remaining 12 determined Bézier–Bernstein coefficients associated with the domain points on T_1. By using (5), it is easy to see that the smoothness conditions across $[v_1,v_3]$ uniquely determines the coefficients $a_{2,1,1}^{[T_2]}$ and $a_{1,2,1}^{[T_2]}$. We have

$$\begin{aligned}a_{1,2,1}^{[T_2]}&=\phi_1^{[T_1]}(v_4)a_{2,2,0}^{[T_1]}+\phi_2^{[T_1]}(v_4)a_{1,3,0}^{[T_1]}+\phi_3^{[T_1]}(v_4)a_{1,2,1}^{[T_1]},\\ a_{2,1,1}^{[T_2]}&=\phi_1^{[T_1]}(v_4)a_{3,1,0}^{[T_1]}+\phi_2^{[T_1]}(v_4)a_{2,2,0}^{[T_1]}+\phi_3^{[T_1]}(v_4)a_{2,1,1}^{[T_1]}.\end{aligned} \tag{11}$$

Finally, the coefficient $a_{1,1,2}^{[T_2]}$ is computed as

$$\begin{aligned}a_{1,1,2}^{[T_2]}=\left(\frac{16}{3}\right)\Bigg[&f\left(\frac{v_1+v_3+2v_4}{4}\right)-\left(\frac{1}{256}\right)(a_{4,0,0}^{[T_2]}+4a_{3,1,0}^{[T_2]}+6a_{2,2,0}^{[T_2]}+4a_{1,3,0}^{[T_2]}+a_{0,4,0}^{[T_2]})\\ &-\left(\frac{1}{32}\right)(a_{3,0,1}^{[T_2]}+3a_{2,1,1}^{[T_2]}+3a_{1,2,1}^{[T_2]}+a_{0,3,1}^{[T_2]})-\left(\frac{3}{32}\right)(a_{2,0,2}^{[T_2]}+a_{0,2,2}^{[T_2]})\\ &-\left(\frac{1}{8}\right)(a_{1,0,3}^{[T_2]}+a_{0,1,3}^{[T_2]})-a_{0,0,4}^{[T_2]}\Bigg],\end{aligned} \tag{12}$$

where we use the remaining interpolation condition. This completes the proof. □

Lemma 6 *($Q \in \mathscr{K}_1$, see Fig. 9, (2)). Let all coefficients $a^{[T_m]}_{i,j,k}$, $m=1,2$, with $i=0$, $j=0$, or $k=0$ and the coefficients $a^{[T_1]}_{1,1,2}$ and $a^{[T_1]}_{2,1,1}$ of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation values $f(P)$, $P \in \mathscr{L}^i_Q \setminus \{(v_1+v_3)/2\}$ uniquely determine the four Bézier–Bernstein coefficients $a^{[T_1]}_{1,2,1}$ and $a^{[T_2]}_{i,j,k}$, $i, j, k \geqslant 1$.*

Proof. Analogously as in (12), the Bézier–Bernstein coefficient $a^{[T_1]}_{1,2,1}$ is uniquely computed by using the interpolation value $f((v_1+2v_2+v_3)/4)$ and the remaining 14 coefficients associated with the domain points on T_1. Then, as in (11), the coefficients $a^{[T_2]}_{2,1,1}$ and $a^{[T_2]}_{1,2,1}$ are uniquely computed by (5), using the C^1 smoothness across $[v_1, v_3]$. Finally, the coefficient $a^{[T_2]}_{1,1,2}$ is uniquely computed, analogously as in (12), by using the interpolation value $f((v_1+v_3+2v_4)/4)$ and the 14 determined coefficients associated with the domain points on T_2. This completes the proof. □

Lemma 7 *($Q \in \mathscr{K}_2$, see Fig. 9, (3)). Let all coefficients $a^{[T_m]}_{i,j,k}$, $m=1,2$, with $i=0$ or $j \leqslant 1$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation value $f((v_1+2v_2+v_3)/4)$ uniquely determines the four Bézier–Bernstein coefficients $a^{[T_2]}_{i,j,k}$, $i \geqslant 1$, $j \geqslant 2$, $m=1,2$.*

Proof. The coefficient $a^{[T_1]}_{2,2,0}$ is uniquely determined by

$$a^{[T_1]}_{2,2,0} = (1/\phi^{[T_1]}_2(v_4))(a^{[T_2]}_{2,1,1} - a^{[T_1]}_{3,1,0}\phi^{[T_1]}_1(v_4) - a^{[T_1]}_{2,2,0}\phi^{[T_1]}_2(v_4)),$$

which results immediately from (5). Analogously, $a^{[T_1]}_{1,3,0}$ is uniquely computed by

$$a^{[T_1]}_{1,3,0} = (1/\phi^{[T_1]}_1(v_4))(a^{[T_2]}_{0,3,1} - a^{[T_1]}_{0,4,0}\phi^{[T_1]}_2(v_4) - a^{[T_1]}_{0,3,1}\phi^{[T_1]}_3(v_4)).$$

Hence, all coefficients associated with the domain points on $[v_1, v_3]$ are determined. Then, the coefficient $a^{[T_1]}_{1,2,1}$ is uniquely computed as in the proof of Lemma 6. Finally, the C^1 smoothness across $[v_1, v_3]$ as in (11) uniquely determines the remaining coefficient $a^{[T_2]}_{1,2,1}$, where we use (5), again. This completes the proof. □

Lemma 8 *($Q \in \mathscr{K}_2$, see Fig. 9, (4)). Let all coefficients $a^{[T_1]}_{i,j,k}$, $i=0$ or $j=0,1$, and $a^{[T_2]}_{i,j,k}$, $i=0,1$ or $j=0$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation value $f((v_1+v_3)/2)$ uniquely determines the three Bézier–Bernstein coefficients $a^{[T_1]}_{1,2,1}$, $a^{[T_1]}_{2,2,0}$ and $a^{[T_2]}_{2,1,1}$.*

Proof. The Bézier–Bernstein coefficient $a^{[T_1]}_{2,2,0}$ is uniquely determined as

$$a^{[T_1]}_{2,2,0} = (\tfrac{1}{6})(16f(P_{2,2,0}) - a^{[T_1]}_{4,0,0} - 4a^{[T_1]}_{3,1,0} - 4a^{[T_1]}_{1,3,0} - a^{[T_1]}_{0,4,0}). \tag{13}$$

Then, the C^1 smoothness across $[v_1, v_3]$ uniquely determines $a^{[T_2]}_{2,1,1}$ as in the proof of Lemma 5. Using (5), we can see that $a^{[T_1]}_{1,2,1}$ is also uniquely determined as

$$a^{[T_1]}_{1,2,1} = \phi^{[T_2]}_1(v_3)a^{[T_2]}_{2,2,0} + \phi^{[T_2]}_2(v_3)a^{[T_2]}_{1,3,0} + \phi^{[T_2]}_3(v_3)a^{[T_2]}_{1,2,1}.$$

This completes the proof. □

In the remaining lemmas, we consider the cases when Q is refined. Here and in the following, we let z be the new vertex in the interior of Q and set $\Delta(Q)=\{T_1, T_2, T_3, T_4\}$, where $T_1=\Delta(z, v_1, v_2)$, $T_2=\Delta(z, v_2, v_3)$, $T_3=\Delta(z, v_3, v_4)$, and $T_4=\Delta(z, v_4, v_1)$.

Lemma 9 *($Q \in \mathscr{K}_2$, see Fig. 9, (5) and (6)). Let all coefficients $a^{[T_m]}_{i,j,k}$, $m=1,2$, $i \leqslant 1$, and $a^{[T_m]}_{i,j,k}$, $m=3,4$, $i=0$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation values $f(P)$, $P \in \mathscr{L}^i_Q$, uniquely determine the 18 Bézier–Bernstein coefficients $a^{[T_m]}_{i,j,k}$, $m=1,3$, $i \geqslant 2$, and $a^{[T_m]}_{i,j,k}$, $m=2,4$, $i \geqslant 1$.*

Proof. The three coefficients $a^{[T_1]}_{4-j,j,0}$, $j=0,1,2$, are uniquely determined by solving a system of linear equations, using the interpolation values $f(P)$, $P\in\{((4-\alpha)z+\alpha v_1)/4: \alpha=0,1,2\}$. This system can be written in the form

$$\begin{pmatrix}1&0&0\\3&4&2\\1&4&6\end{pmatrix}\cdot x=y, \tag{14}$$

where $x=(a^{[T_1]}_{4,0,0},a^{[T_1]}_{3,1,0},a^{[T_1]}_{2,2,0})^t$ and $y=(y_1,y_1,y_3)$ is given by

$$y_1=f(z),$$
$$y_2=\left(\frac{256}{27}\right)f\left(\frac{3z+v_1}{4}\right)-\left(\frac{4}{9}\right)a^{[T_1]}_{1,3,0}+\left(\frac{1}{27}\right)a^{[T_1]}_{0,4,0},$$
$$y_3=16f\left(\frac{3z+v_1}{4}\right)-4a^{[T_1]}_{1,3,0}-a^{[T_1]}_{0,4,0}.$$

The coefficient $a^{[T_1]}_{2,0,2}$ is computed using (5):

$$a^{[T_1]}_{2,0,2}=(1/\phi^{[T_1]}_1(v_3))(a^{[T_2]}_{1,1,2}-a^{[T_1]}_{1,1,2}\phi^{[T_1]}_2(v_3)).$$

Now, analogously as in (14), the coefficient $a^{[T_1]}_{3,0,1}$ is uniquely determined by $a^{[T_1]}_{4,0,0}$, $a^{[T_1]}_{j,0,4-j}$, $j=0,\dots,2$, and the value $f((3z+v_2)/4)$. Then, the coefficient $a^{[T_1]}_{2,1,1}$ is computed as in (12), using the value $f((z+v_1+v_2)/4)$ and the remaining 14 Bézier–Bernstein coefficients associated with the domain points in T_1. Now, the coefficients $a^{[T_4]}_{3-j,1,j}$, $j=0,\dots,2$, are uniquely determined by using (5):

$$a^{[T_4]}_{3-j,1,j}=a^{[T_1]}_{4-j,0,j}\phi^{[T_1]}_1(v_4)+a^{[T_1]}_{3-j,0,1+j}\phi^{[T_1]}_2(v_4)+a^{[T_1]}_{3-j,1,j}\phi^{[T_1]}_3(v_4),\quad j=0,\dots,2.$$

(Note, that $\phi^{[T_1]}_3(v_4)$ is zero, if and only if $[z,v_1]$ is degenerate at z.) Moreover, $a^{[T_4]}_{1,3,0}$ is computed as

$$a^{[T_4]}_{1,3,0}=(1/\phi^{[T_4]}_1(v_3))(a^{[T_3]}_{0,1,3}-a^{[T_2]}_{0,4,0}\phi^{[T_2]}_2(v_3)-a^{[T_4]}_{0,3,1}\phi^{[T_2]}_3(v_3)).$$

(Note, that again $\phi^{[T_2]}_2(v_4)$ is zero, if and only if $[z,v_1]$ is degenerate at z.) The coefficient $a^{[T_4]}_{1,1,2}$ is computed as in (12), using the value $f((z+2v_4+v_1)/4)$ and the remaining 14 Bézier–Bernstein coefficients associated with the domain points in T_4. Now, we consider Bézier–Bernstein coefficients associated with domain points in T_2 and T_3. Using (5), the coefficients $a^{[T_2]}_{2+j,1-j,1}$, $j\leqslant 1$, and $a^{[T_3]}_{2-j,1,1+j}$, $j\leqslant 1$, are uniquely determined as (5):

$$a^{[T_2]}_{2+j,1-j,1}=a^{[T_1]}_{3+j,1-j,0}\phi^{[T_1]}_1(v_3)+a^{[T_1]}_{2+j,1-j,1}\phi^{[T_1]}_2(v_3),\quad j=0,1,$$
$$a^{[T_3]}_{2-j,1,1+j}=a^{[T_4]}_{3-j,0,1+j}\phi^{[T_4]}_1(v_3)+a^{[T_4]}_{2-j,1,1+j}\phi^{[T_4]}_2(v_3),\quad j=0,1.$$

If Q is convex, then analogously as in (13), the coefficient $a^{[T_2]}_{2,0,2}$ is computed by the interpolation value $f((z+v_3)/2)$ and the four determined Bézier–Bernstein coefficients associated with the domain points on $[z,v_2]$. If Q is non-convex, the formula in (5) yields

$$a^{[T_2]}_{2,0,2}=(1/\phi^{[T_2]}_3(v_4))(a^{[T_3]}_{2,1,1}-a^{[T_2]}_{3,0,1}\phi^{[T_2]}_1(v_4)-a^{[T_2]}_{2,1,1}\phi^{[T_2]}_2(v_4)).$$

Finally, $a^{[T_3]}_{1,2,1}$ is uniquely determined and computed by (5):

$$a^{[T_3]}_{1,2,1}=a^{[T_2]}_{2,2,0}\phi^{[T_2]}_1(v_4)+a^{[T_2]}_{1,1,2}\phi^{[T_2]}_2(v_4)+a^{[T_2]}_{1,0,3}\phi^{[T_2]}_3(v_4),$$

where $\phi^{[T_2]}_3(v_4)$ is zero, if and only if $[z,v_3]$ is degenerate at z. The proof is complete. □

Lemma 10 (*$Q \in \mathscr{K}_3$, see Fig. 9, (7) and (8)). Let all coefficients $a^{[T_m]}_{i,j,k}$, $m = 1, \ldots, 3$, $i \leqslant 1$, and $a^{[T_4]}_{i,j,k}$, $m = 4$, $i = 0$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation values $f(P)$, $P \in \mathscr{L}^i_Q$, uniquely determine the 15 Bézier–Bernstein coefficients $a^{[T_m]}_{i,j,k}$, $i \geqslant 2$, $m = 1, \ldots, 3$, and $a^{[T_4]}_{i,j,k}$, $i \geqslant 1$.*

Proof. Except for $a^{[T_1]}_{2,2,0}$, each of the unknown Bézier–Bernstein coefficient is uniquely determined and computed by using analogous arguments as in the proof of Lemma 9. From (5), we obtain

$$a^{[T_1]}_{2,2,0} = (1/\phi^{[T_1]}_1(v_4))(a^{[T_4]}_{1,1,2} - a^{[T_1]}_{1,3,0}\phi^{[T_1]}_2(v_4) - a^{[T_1]}_{1,2,1}\phi^{[T_1]}_3(v_4)).$$

The proof is complete. □

Lemma 11 (*$Q \in \mathscr{K}_4$, see Fig. 9, (9)). Let Q be convex and all coefficients $a^{[T_m]}_{i,j,k}$, $m = 1, \ldots, 4$, $i \leqslant 1$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation values $f(P)$, $P \in \mathscr{L}^i_Q$, uniquely determine the 13 Bézier–Bernstein coefficients $a^{[T_m]}_{i,j,k}$, $i \geqslant 2$, $m = 1, \ldots, 4$.*

Proof. Except for $a^{[T_1]}_{2,2,0}$ and $a^{[T_4]}_{2,2,0}$, each unknown Bézier–Bernstein coefficient is uniquely determined and computed by using analogous arguments as in the proof of Lemma 9. By (5), we obtain for these coefficients

$$a^{[T_1]}_{2,2,0} = (1/\phi^{[T_1]}_1(v_4))(a^{[T_4]}_{1,1,2} - a^{[T_1]}_{1,3,0}\phi^{[T_1]}_2(v_4) - a^{[T_1]}_{1,2,1}\phi^{[T_1]}_3(v_4)),$$
$$a^{[T_4]}_{2,2,0} = (1/\phi^{[T_4]}_1(v_3))(a^{[T_3]}_{1,1,2} - a^{[T_4]}_{1,2,1}\phi^{[T_4]}_3(v_3)).$$

The proof is complete. □

Lemma 12 (*$Q \in \mathscr{K}_4$, see Fig. 9, (10)). Let Q be non-convex and all coefficients $a^{[T_m]}_{i,j,k}$, $m = 1, \ldots, 4$, $i \leqslant 1$, of a spline $s \in \mathscr{S}^1_4(\Delta(Q))$ be determined. Then, the interpolation values $f(P)$, $P \in \mathscr{L}^i_Q$, uniquely determine the 13 Bézier–Bernstein coefficients $a^{[T_m]}_{i,j,k}$, $i \geqslant 2$, $m = 1, \ldots, 4$.*

Proof. It can be seen analogously as in the proof of Lemma 11 that the coefficients $a^{[T_m]}_{2,2,0}$, $m = 1, \ldots, 4$, $a^{[T_1]}_{4,0,0}$ and $a^{[T_1]}_{3,0,1}$ are uniquely determined. The remaining seven Bézier–Bernstein coefficients are computed by solving a system of linear equations. To explain this, we introduce the following notation for the quadrangle Q (see Fig. 11): $z = (0, 0)$, $v_1 := (0, -\hat{x})$, $v_2 := (\tilde{y}, m\tilde{y})$, $v_3 := (0, \hat{x})$, and $v_4 = (\tilde{x}, n\tilde{x})$, where $\hat{x}, \tilde{x} > 0$, $\tilde{y} < 0$, and $m, n \in \mathbb{R}$. Since $[z, v_2]$ and $[z, v_4]$ are not parallel, we have $m \neq n$.

Furthermore, let x_ℓ, $\ell = 1, \ldots, 5$, be the unknown coefficients, i.e., we set

$$x = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{t}} = (a^{[T_1]}_{2,1,1}, a^{[T_3]}_{2,1,1}, a^{[T_3]}_{3,0,1}, a^{[T_4]}_{2,1,1}, a^{[T_4]}_{3,0,1})^{\mathrm{t}}.$$

In the following, we consider the homogeneous problem and show that only the trivial solution $(x_1, \ldots, x_5) = 0$ satisfies all C^1 smoothness conditions of $\mathscr{S}^1_4(\Delta(Q))$. From (5), we obtain $a^{[T_2]}_{2,1,1} = -a^{[T_1]}_{2,1,1} = -x_1$, $a^{[T_2]}_{3,0,1} = -a^{[T_1]}_{3,1,0} = -x_5$, and

$$-x_1 = \phi^{[T_3]}_1(v_2) \cdot -x_5 + \phi^{[T_3]}_2(v_2) \cdot 0 + \phi^{[T_3]}_3(v_2) \cdot x_2,$$
$$x_4 = \phi^{[T_3]}_1(v_1) \cdot x_3 + \phi^{[T_3]}_2(v_1) \cdot x_2 + \phi^{[T_3]}_3(v_1) \cdot 0,$$
$$x_1 = \phi^{[T_4]}_1(v_2) \cdot x_5 + \phi^{[T_4]}_2(v_2) \cdot x_4 + \phi^{[T_4]}_3(v_2) \cdot 0,$$
$$x_3 = \phi^{[T_1]}_1(v_4) \cdot 0 + \phi^{[T_1]}_2(v_4) \cdot x_5 + \phi^{[T_1]}_3(v_4) \cdot 0,$$

The interpolation condition at that point $P = (2z + v_1 + v_2)/4$ in the interior of T_1 and (1) imply

$$0 = 4(\phi^{[T_1]}_1(P))^3 \cdot \phi^{[T_1]}_2(P) \cdot x_5 + 12(\phi^{[T_1]}_1(P))^2 \cdot \phi^{[T_1]}_2(P) \cdot \phi^{[T_1]}_3(P) \cdot x_1.$$

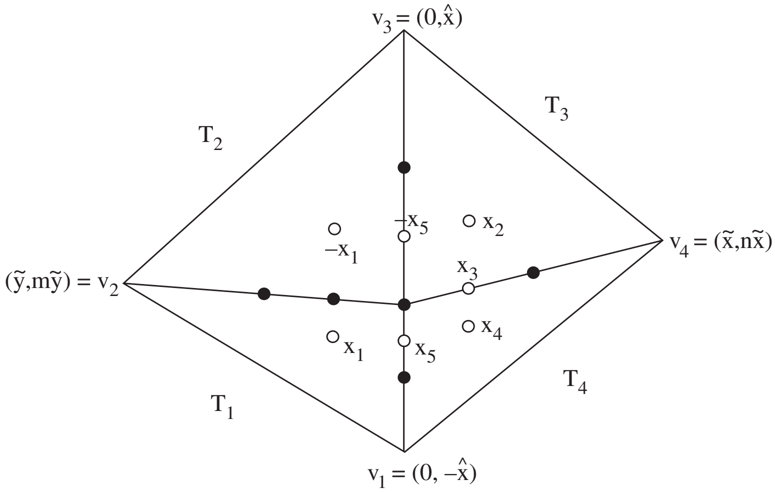

Fig. 11. Structure of Q and notations used in the proof of Lemma 12. For better illustration Q is drawn convex in the figure (and therefore $z = (0, 0)$ is not shown as the midpoint of the line segment between $v_1 = (0, -\hat{x})$ and $v_3 = (0, \hat{x})$), while the lemma deals with the non-convex case.

Due to the structure of Q, the barycentric coordinates appearing above are computed as follows:

$$\phi_3^{[T_3]}(v_2) = \phi_3^{[T_4]}(v_2) = \frac{\tilde{y}}{\tilde{x}}, \quad \phi_1^{[T_3]}(v_1) = 2, \quad \phi_2^{[T_1]}(v_4) = \frac{\tilde{x}}{\hat{x}}(m-n),$$
$$\phi_1^{[T_4]}(v_2) = 1 - \frac{\tilde{y}}{\tilde{x}} - \frac{\tilde{y}}{\hat{x}}(n-m), \quad \phi_1^{[T_3]}(v_2) = 1 - \frac{\tilde{y}}{\tilde{x}} - \frac{\tilde{y}}{\hat{x}}(m-n),$$
$$\phi_2^{[T_3]}(v_1) = -1, \quad \phi_1^{[T_1]}(P) = \tfrac{1}{2}, \quad \phi_2^{[T_1]}(P) = \phi_3^{[T_1]}(P) = \tfrac{1}{4}.$$

This leads to the system

$$\begin{pmatrix} 1 & \frac{\tilde{y}}{\tilde{x}} & 0 & 0 & -1 + \frac{\tilde{y}}{\tilde{x}} + \frac{\tilde{y}}{\hat{x}}(m-n)) \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & \frac{\tilde{y}}{\tilde{x}} & 1 - \frac{\tilde{y}}{\tilde{x}} - \frac{\tilde{y}}{\hat{x}}(n-m) \\ 0 & 0 & -1 & 0 & \frac{\tilde{x}}{\hat{x}}(m-n) \\ \frac{3}{64} & 0 & 0 & 0 & \frac{1}{64} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0. \tag{15}$$

We compute the determinant D of the 5×5 matrix corresponding to this system as

$$D = \frac{3\tilde{y}^2}{16\tilde{x}\hat{x}}(m-n) \neq 0.$$

Therefore, given arbitrary interpolation values all Bézier–Bernstein coefficients $a_{i,j,k}^{[T_m]}$, $i \geqslant 2$, $m = 1, \ldots, 4$, are uniquely determined. This completes the proof. □

6. Proof of main result and error bound

In this section, we prove the main results of our paper given in Theorem 4. Moreover, we establish an error bound for our interpolation method.

Proof of Theorem 4. Given $f \in C(\Omega)$, we show how to uniquely compute the Bézier Bernstein coefficients of the interpolating spline s. Simultaneously we show that this computation is local and stable. We do this by first considering the spline along the edges of the quadrangles and use Algorithm 3 in connection with the choice of interpolation points at the edges. Once we have shown that the spline s is uniquely determined along the edges, we consider the classification resulting from Algorithm 2 which is based on the decomposition of Algorithm 1. Using this classification and the interpolation conditions in the interior, we show that the spline is uniquely and locally determined in the interior.

We begin by considering the edges of the quadrangles. Let $Q \in \mathscr{N}_0$, where $\mathscr{N}_0$ is the first class of quadrangles determined by Algorithm 3. The Bézier Bernstein coefficients of s that are associated with the domain points on a boundary edge e of Q are uniquely determined by the interpolation conditions at the interpolation points on e. This is univariate, polynomial interpolation. More precisely, the coefficients are computed by solving a system of five linear equations. Fixing an appropriate triangle T in Q, this system can be written in the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 81 & 108 & 54 & 12 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 12 & 54 & 108 & 81 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = y, \tag{16}$$

where $x = (a^{[T]}_{4,0,0}, a^{[T]}_{3,1,0}, a^{[T]}_{2,2,0}, a^{[T]}_{1,3,0}, a^{[T]}_{0,4,0})^{\mathrm{t}}$ and $y \in \mathbb{R}^5$ is determined by the values of f in $P^{[T]}_{4,0,0}$, $P^{[T]}_{3,1,0}$, $P^{[T]}_{2,2,0}$, $P^{[T]}_{1,3,0}$, and $P^{[T]}_{0,4,0}$. Hence, these coefficients depend only on the data values of f from points in $st(v)$, where v is a suitable vertex, and in (4) the constant becomes $C = \|M^{-1}\|$, where M is the matrix on the left side of (16). The same argument can be applied to each of the boundary edges of the quadrangles in $\mathscr{N}_0$. Since the quadrangles in $\mathscr{N}_0$ do not have common vertices (Lemma 3, (i)), all these coefficients are determined independently.

We proceed by considering the quadrangles from the classes $\mathscr{N}_1$, $\mathscr{N}_2$, $\mathscr{N}_3$, and $\mathscr{N}_4$ (in this order). Since we can now apply a similar argument as in Nürnberger et al. [22] (see also [19]), we can be relatively brief, here. The Bézier–Bernstein coefficients associated with the domain points on the boundary edges e of quadrangles from these classes (as well as those on the diagonal of the quadrangles in Δ^1_Q) are now uniquely determined by the interpolation conditions on e and the C^1 smoothness at the endpoints of e being vertices of a quadrangle in a class with lower index. For each class, we determine these Bézier–Bernstein coefficients by solving a system of linear equations, which can be written in the form $M_1 x = y$, where M_1 is a submatrix of M, and x, and y are similar as in (16). Note that Lemma 3 (ii) implies that for each class $\mathscr{N}_j$ all these coefficients are determined independently. In particular, there is no propagation within each of these classes, because the method guarantees that propagation is only possible from lower to higher indexed classes. It follows that these coefficients associated with the domain points on the boundary edges e of quadrangles from $\mathscr{N}_1$, $\mathscr{N}_2$, $\mathscr{N}_3$, and $\mathscr{N}_4$ depend only on the data values of f from points in $st^n(v)$, where v is a suitable vertex, and $n = 3$, 5, 7, and 9, respectively. Moreover, we note that involving C^1 smoothness at the endpoints leads to stable computations depending only on the smallest angle of $\widetilde{\Delta}$. Hence, at this point, we conclude that for the coefficients of consideration (4) holds with an appropriate constant C.

Now, we consider the remaining Bézier–Bernstein coefficients of s associated with domain points in the interior. For each triangle $T \in \Delta_T$, the three interpolation conditions involving the values $f(P)$, $P \in \mathscr{L}^i_T$, uniquely determine the Bézier–Bernstein coefficients $a^{[T]}_{2,1,1}$, $a^{[T]}_{1,2,1}$, and $a^{[T]}_{1,1,2}$. These coefficients are computed by solving a system of three linear equations, which is analogous to the system in (10) appearing in the proof of Lemma 5. In view of Lemma 2(i), for each of these triangles these coefficients are determined independently. It follows that these coefficients depend only on the data values of f from points in $st^9(v)$, where v is a suitable vertex, and we obtain (4) with a constant C depending on $\|M_0^{-1}\|$, where M_0 is the matrix on the left side of (10). Next, we consider the quadrangles in $\Delta^1_Q \cup \Delta^2_Q$, where we take into account the class $\mathscr{K}_i$, $i = 0, \ldots, 4$, to which the quadrangle belongs as well as the order given by Algorithm 2. Again, we emphasize that the interpolation method guarantees that propagation is only possible from lower to higher indexed classes, and not within a class. In the following, we briefly call the edges of quadrangles already considered C^1 *determined edges*. Let $Q \in \mathscr{K}_0$. It follows from the above that all Bézier–Bernstein coefficients associated with the domain points on the edges of Q are already uniquely determined. Hence, the assumptions of Lemma 5 are satisfied. Therefore, Lemma 5 together with the interpolation conditions imposed for points in the interior of Q imply that all coefficients associated with the domain points in the interior of Q are uniquely determined. In view of Lemma 2(i), again, these coefficients depend only on the data values of f from points in $st^9(v)$, where v is a suitable vertex, and we obtain

(4) with a constant C depending on $\|M_0^{-1}\|$, where M_0 is the matrix on the left side of (10). Now, let $Q \in \mathscr{K}_1$. Then, exactly one edge e of Q is C^1 determined. Let $e=[v_1, v_3]$. Now, using (5), the Bézier–Bernstein coefficients $a_{1,1,2}^{[T_1]}$ and $a_{2,1,1}^{[T_1]}$ are uniquely determined. Here, we use that the Bézier–Bernstein coefficients associated with the domain points in the neighboring quadrangle of $\Delta_T \cup \mathscr{K}_0$ are already determined. In view of Lemma 2(iii) (case $j=1$), it follows that these coefficients depend only on the data values of f from points in $st^9(v)$, where v is a suitable vertex. Since we use (6) here, these computations are also stable involving only the smallest angle of $\widetilde{\Delta}$. Now, the suppositions of Lemma 6 are satisfied. Therefore, it follows from the remaining interpolation conditions that all coefficients associated with the domain points in the interior of Q are uniquely determined. It follows that these coefficients depend only on the data values of f from points in $st^{10}(v)$, where v is a suitable vertex, and the corresponding constant C in (4) involves the maximum of all constants appearing in (6) when following the proof of Lemma 6 as well as the smallest angle of $\widetilde{\Delta}$. Now, let $Q \in \mathscr{K}_2$. In all three cases (see third, fourth, and fifth quadrangle from the left in Fig. 6), the Bézier–Bernstein coefficients associated with the domain points on the edges of Q are already determined. Since s is already uniquely determined on the neighboring triangles and quadrangles in $\Delta_T \cup \mathscr{K}_0 \cup \mathscr{K}_1$, it follows from the C^1 smoothness (5) that some Bézier Bernstein coefficients associated with the domain points in the interior of Q are determined. In what follows we describe these coefficients more precisely. If the edges $[v_1, v_3]$ and $[v_1, v_4]$ are C^1 determined, these coefficients are $a_{1,j,k}^{[T_m]}$, $m=1,2$. Otherwise, if the edges $[v_1, v_3]$ and $[v_2, v_4]$ are C^1 determined, these coefficients are $a_{i,1,k}^{[T_1]}$, and $a_{1,j,k}^{[T_2]}$. Finally, if the edges $[v_1, v_3]$ and $[v_2, v_3]$ are C^1 determined, these coefficients are $a_{1,j,k}^{[T_m]}$, $m=1,3$. The corresponding computations are stable, since (6) and (7) hold. Moreover, it follows that these coefficients depend only on the data values of f from points in $st^{11}(v)$, where v is a suitable vertex. Now, the suppositions of Lemma 7, 8, and 9, respectively, are satisfied. Therefore, it follows from the interpolation conditions that all coefficients associated with the domain points in the interior of Q are uniquely determined. We conclude that these coefficients depend only on the data values of f from points in $st^{12}(v)$, where v is a suitable vertex, and the corresponding constant C in (4) involves the maximum of $\|M_2^{-1}\|$, (M_2 being the matrix on the left side of (14)) and all constants appearing in Eqs. (6) and (7) when following the proof of the Lemma 7, 8, and 9, respectively. Note that we use Lemma 2(iii) (case $j=2$), here. Now, let $Q \in \mathscr{K}_3$. Then, exactly three edges of Q are C^1 determined. Let $[v_1, v_3]$, $[v_1, v_4]$, and $[v_2, v_3]$ be these edges. Hence, using (5), the coefficients $a_{i,j,k}^{[T_m]}$, $i \leqslant 1$, $m=1,\ldots,3$, are uniquely determined since s is already determined on the neighboring triangles and quadrangles in $\Delta_T \cup \mathscr{K}_0 \cup \ldots \cup \mathscr{K}_2$. Since (6) holds, this computation is stable and these coefficients depend only on the data values of f from points in $st^{12}(v)$, where v is a suitable vertex. Now, the suppositions of Lemma 10 are satisfied. Therefore, the interpolation conditions imply that all coefficients associated with the domain points in the interior of Q are uniquely determined. Hence, these coefficients depend only on the data values of f from points in $st^{13}(v)$, where v is a suitable vertex, and the corresponding constant C in (4) involves the maximum of all constants appearing in Eqs. (6) and (7) when following the proof of Lemma 10, and therefore the smallest angle of $\widetilde{\Delta}$. Note that we use Lemma 2(iii) (case $j=3$), here. Finally, let $Q \in \mathscr{K}_4$. Since all edges of Q are C^1 determined, all coefficients $a_{i,j,k}^{[T_m]}$, $i \leqslant 1$, $m=1,\ldots,4$, are uniquely determined by using (5). It follows that these coefficients depend only on the data values of f from points in $st^{13}(v)$, where v is a suitable vertex, and moreover these computations are stable. Now, the suppositions of Lemmas 11 and 12, respectively, are satisfied. Therefore, it follows from the interpolation conditions that all coefficients associated with the domain points in the interior of Q are uniquely determined. Hence, these coefficients depend only on the data values of f from points in $st^{14}(v)$, where v is a suitable vertex, and the corresponding constant C in (4) involves the maximum of $\|M_4^{-1}\|$, where M_4 is the matrix on the left side of (15)), and all constants appearing in Eqs. (6) and (7) when following the proof of the lemmas, and therefore the smallest angle of $\widetilde{\Delta}$. Again, we use Lemma 2(iii) (case $j=4$), here.

We conclude that all Bernstein Bèzier coefficients of s are uniquely determined from the interpolation conditions. Each coefficient depends only on the data values of f from points in $st^{14}(v)$, where v is a suitable vertex. Moreover, we have shown that the corresponding constant C in (4) involves the norm of a fixed number of matrices as well as the smallest angle of $\widetilde{\Delta}$, but no further geometric properties of $\widetilde{\Delta}$. In connection with the proof of the below Theorem 13, the proof is complete. $\square$

The arguments in the proof of Theorem 4 are based on the priority principles resulting from Algorithm 2 and 3, and show that the value of $f(P)$ at an interpolation point P may influence the computation of the Bézier Bernstein coefficients associated with domain points in $st^{14}(v)$. However, according to our experience this might happen in rare cases, only. Another example for this kind of propagation is given in Fig. 12.

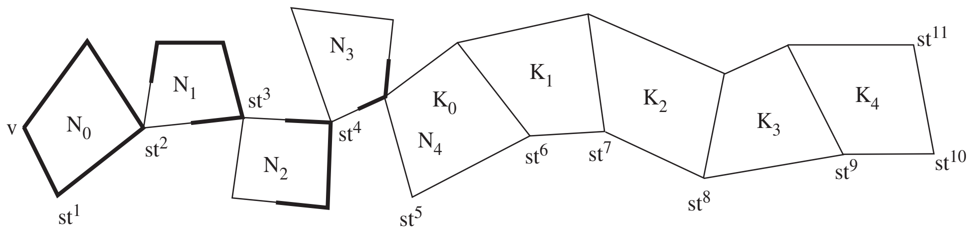

Fig. 12. Propagation of an interpolation value. In this somewhat extreme example, the interpolation value $f(v)$ at v may have an influence on the computation of some of the Bézier–Bernstein coefficients in $st^{11}(v)$. In this illustration, each N_j, $j = 0, \ldots, 4$, represents a quadrangle in the class $\mathscr{N}_j$ and each K_j, $j = 0, \ldots, 4$, represents a quadrangle in the class $\mathscr{K}_j$.

In the following, we give bounds on the error of our interpolation method, which show that the interpolating splines yield optimal approximation order. To do this, we denote partial derivatives by $D_x^i D_y^j$ and let $|\cdot|_{5,\Omega}$ be the usual Sobolev semi-norm defined for functions from the Sobolev space $W_\infty^5(\Omega)$. In addition, for any piecewise polynomial function $\tilde{s}$ on $\tilde{\Delta}$, we let $\|\tilde{s}\|_B$ be the maximum of the infinity norms of $\tilde{s}$ over the triangles of a triangulation $\Delta_B \subseteq \tilde{\Delta}$. Furthermore, we let h be the maximal diameter of the triangles in $\widetilde{\Delta}$ and $s_f \in \mathscr{S}_4^1(\widetilde{\Delta})$ be the interpolating spline of $f \in W_\infty^5(\Omega)$ corresponding to our method.

Theorem 13. *There exists a constant K depending only on the smallest angle in* Δ, *such that*

$$\|D_x^i D_y^j (f - s_f)\|_\Omega \leqslant K h^{5-i-j} |f|_{5,\Omega},$$

where $0 \leqslant i + j \leqslant 4$.

Proof. The proof of this result is similar to the proof of Theorem 7.1 in Nürnberger et al. [19]. Let T be a triangle of $\widetilde{\Delta}$ and set $B := st^{14}(v)$, where v is an appropriate vertex of T. Then, it is well known that there exists a polynomial $q_f \in \mathscr{P}_4$ such that

$$\|D_x^i D_y^j (f - q_f)\|_B \leqslant K_0 H^{5-i-j} |f|_{5,B},$$

where H is the diameter of B, K_0 is a constant depending only on the smallest angle of the triangles in B, and $0 \leqslant i + j \leqslant 4$. Since $H \leqslant 28h$, it follows that

$$\|D_x^i D_y^j (f - q_f)\|_B \leqslant K_1 \cdot h^{5-i-j} |f|_{5,B}, \tag{17}$$

with $K_1 := 28^5 \cdot K_0$, where $0 \leqslant i + j \leqslant 4$. Since $s_{q_f} = q_f$, we obtain

$$\|D_x^i D_y^j (f - s_f)\|_T \leqslant \|D_x^i D_y^j (f - q_f)\|_T + \|D_x^i D_y^j (q_f - s_f)\|_T, \tag{18}$$

and hence it suffices to estimate the second term because of (17). The Markov inequality (see [29]) yields

$$\|D_x^i D_y^j (q_f - s_f)\|_T \leqslant K_2 \cdot h^{-i-j} \|q_f - s_f\|_T,$$

where K_2 depends on the smallest angle in T. The polynomial piece $(q_f - s_f)|_T \in \mathscr{P}_4$ can be written in its Bézier Bernstein form as

$$q_f - s_f = \sum_{i+j+k=4} a_{i,j,k}^{[T]} B_{i,j,k}.$$

The Bernstein polynomials form a partition of unity. In particular, they are non-negative on T. Hence,

$$\|q_f - s_f\|_T \leqslant \max_{P_{i,j,k}^{[T]} \in D_T} |a_{i,j,k}^{[T]}|. \tag{19}$$

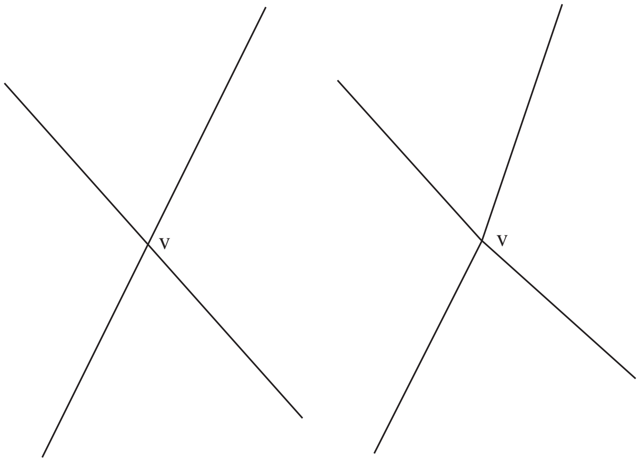

Fig. 13. Singular vertex (left) and near-singular vertex (right).

Now, we use Theorem 4. It follows from (4), where $\mathscr{L}_{i,j,k}^{[T]} \subseteq B \cap \mathscr{L}$, that for all $P_{i,j,k}^{[T]} \in D_T$

$$|a_{i,j,k}^{[T]}| \leqslant C \cdot \max_{z \in B \cap \mathscr{L}} |(f - q_f)(z)| \leqslant C \cdot \|f - q_f\|_B,$$

where C is a constant depending only on the smallest angle $\tilde{\alpha}$ in $\widetilde{\Delta}$. Inserting this in (19) and then in (18), we obtain

$$\|D_x^i D_y^j (f - s_f)\|_T \leqslant \widetilde{K} \cdot h^{5-i-j} |f|_{5,B},$$

with $\widetilde{K} = K_1(1 + CK_2)$, which immediately implies an estimate of the desired type with a constant $\widetilde{K}$ depending only on the smallest angle $\tilde{\alpha}$ in $\widetilde{\Delta}$. To show the assertion with a constant K depending on the smallest angle α in Δ, it now suffices to show that $\tilde{\alpha} \geqslant (\frac{1}{3})\alpha$. The refinement $\tilde{\Delta}$ is obtained by adding edges to some of the quadrangles (see Section 3.3). Moreover, in some exceptional cases (see Fig. 3, right) we apply Clough–Tocher splits. In this case, the above relation between the angles $\tilde{\alpha}$ and α has been shown in Lai and Schumaker [17]. According to the remark in Section 3.3, it now suffices to consider a non-convex quadrangle for which we add edges. In case of a refined non-convex quadrangle, we let $\overline{v} = (v_1 + v_2 + v_3)/3$ be the barycenter of T_1, where we use the notations in Fig. 9. Moreover, let P be the intersection point of the lines ℓ_1, ℓ_2 through$v_2, \overline{v}$ and v_1, v_3, respectively. In order to prove $\tilde{\alpha} \geqslant (\frac{1}{3})\alpha$ in this situation, we show that $P = z$, where z is the new vertex included in Q. Since $P \in \ell_1$, we have $P = \beta v_2 + (1 - \beta)\overline{v}$ for a suitable β. It follows from the definition of $\overline{v}$ that $P = v_2(\beta + (1 - \beta)/3) + v_1(1 - \beta)/3 + v_3(1 - \beta)/3$ and hence, because of $P \in \ell_2$, $\beta = -1/2$. Thus, $P = (v_1 + v_3)/2 = z$. This proof of the theorem is complete. $\square$

Remark. In Lai and Schumaker [16] interior vertices v of Δ were called *near-singular*, if exactly four edges of Δ emanate at v, and *each* of these edges is near-degenerate at v. Moreover, v is called *singular* if the four edges have exactly two different slopes (see Fig. 13). A possible extension of our decomposition method is to regard cells of singular and near-singular vertices as elements of Δ_Q^1 and Δ_Q^2 from the beginning of the algorithm. If this is done, special triangulations like the four-directional mesh or other triangulations with a lot of singular vertices would have certain advantageous properties for the resulting quartic C^1 splines. On the other hand, it is well known that near-singular vertices can sometimes lead to unstable computations of the interpolating spline. In order to keep our decomposition of triangulations simple while treating near-singular vertices properly, we described the simpler approach, here.

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