From CONTINUOUS WAVELET TRANSFORM To WAVELET BASES

Discretizing CWT

$$(Wf)(a,b) = a^{-1/2} \int f(t)\psi\left(\frac{t-b}{a}\right) dt$$

$a > 0, b \in R$, where

$$\int |\widehat{\psi}(\omega)|^2 \omega^{-1} d\omega < \infty$$

(admissibility conditiion)

- CWT represents an ONE-variable function f by a TWO-variable function Wf $\longrightarrow$ **redundancy**

- Can we discretize the CWT without loosing any information about f?

Some basic mathematical notations and definitions

$L_2(R) = \{f : \int_R f(t)^2 dt < \infty\}$ (functions of finite energy)

$L_2(R)$ is a Hilbert space with the inner product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(t) f_2(t) dt$$

$$||f||^2 = (f, f) = \int_{-\infty}^{\infty} f(t)^2 dt$$

Definition 1 *A sequence of functions $\phi_1, \phi_2, ...$ from $L_2(R)$ is a Riesz basis in $L_2(R)$ if*

1. *a linear span of $\phi_1, ...,$ is a dense in $L_2(R)$*

2. *there exist $0 < A \leq B < \infty$ such that*

$$A \sum_j c_j^2 \leq || \sum_j c_j \phi_j ||^2 \leq B \sum_j c_j^2$$

for all square-summable sequences $\{c_j\}$

Theorem 1 *If $\{\phi_j\}$ is a basis in $L_2(R)$ then one can approximate any $f \in L^2(R)$ with any given accuracy by a finite linear combination of basis functions, i.e. for any given ε one can find sufficiently large m and coefficients $a_1, ..., a_m$ such that*

$$||f - \sum_{j=0}^{m} a_j \phi_j|| < \varepsilon$$

Definition 2 *A basis $\{\phi_j\}$ is orthonormal if $(\phi_i, \phi_j) = \delta_{ji}$.*

Examples: Hermite functions

For orthonormal bases:

- $f(t) = \sum_j (f, \phi_j) \phi_j(t)$
- $||f||^2 = \sum_j |(f, \phi_j)|^2$ (Parseval's identity)

Definition 3 *A linear subspace* $\mathcal{L}$ *of* $L_2(R)$ *is a closed subspace if it contains all of its limiting points, i.e.*

$$||f - f_m|| \to 0,\ f_m \in \mathcal{L} \longrightarrow f \in \mathcal{L}$$

Theorem 2 *(Projection Theorem). Let* $\mathcal{L}$ *be a closed subspace in* L^2 *and* $f \in L_2$*. Then,*

1. *there exists a unique* $P_{\mathcal{L}}f \in \mathcal{L}$ *such that*

$$||f - P_{\mathcal{L}}f|| = \inf_{g \in \mathcal{L}} ||f - g||$$

2. $P_{\mathcal{L}}f \in \mathcal{L}$ *and* $||f - P_{\mathcal{L}}f|| = \inf_{g \in \mathcal{L}} ||f - g||$ *iff* $P_{\mathcal{L}}f \in \mathcal{L}$ *and* $f - P_{\mathcal{L}}f \in \mathcal{L}^{\perp}$

$L^2 = \mathcal{L} \oplus \mathcal{L}^{\perp}$, where $\mathcal{L}^{\perp}$ is an orthogonal complement of $\mathcal{L}$

Example: $\mathcal{L}$ - even functions in L^2, $\mathcal{L}^{\perp}$ - odd functions in L^2

$$a = a_0^{-j}, a_0 \neq 1;\ b = kb_0a_0^{-j} \quad (j, k \in Z)$$

$$\psi_{jk}(t) = a_0^{j/2}\psi\left(\frac{t - kb_0a_0^{-j}}{a_0^{-j}}\right) = a_0^{j/2}\psi(a_0^j t - kb_0)$$

Discrete wavelet coefficients:

$$(f, \psi_{jk}) = \int f(t)\psi_{jk}(t)dt$$

- Do the discrete wavelet coefficients completely characterize f?

  $$(f_1, \psi_{jk}) = (f_2, \psi_{jk}), \forall j, k \in Z \ \longrightarrow\ f_1 \equiv f_2$$

  or, equivalently,

  $$(f, \psi_{jk}) = 0, \forall j, k \in Z \ \longrightarrow\ f \equiv 0$$

- Can any function $f \in L_2$ be represented as a superposition of atoms ψ_{jk}?

The set of $\psi_{jk},\ j,k \in Z$ should constitute a *frame*, i.e. there should exist the constants $A, B > 0$ such that for any $f \in L_2$

$$A||f||^2 \leq \sum_{j,k} |(f, \psi_{jk})|^2 \leq B||f||^2$$

- frame is generally an *overcomplete* set of atoms since ψ_{jk} can be linearly dependent

- Riesz basis is a frame whose vectors are linearly independent.

Example.
Let $e_1 = (0,1), e_2 = (-\frac{\sqrt{3}}{2}, -1/2), e_3 = (\frac{\sqrt{3}}{2}, -1/2)$.
Then for any $v = (v_1, v_2)$

$$\sum_{j=1}^{3} |(v, e_j)|^2 = ... = \frac{3}{2}(v_1^2 + v_2^2) = \frac{3}{2}||v||^2,$$

$A = B = 3/2 \longrightarrow \{e_1, e_2, e_3\}$ is a *tight* frame, but not a basis (linearly dependent!)

Which choices for ψ, a_0, b_0 will lead to frames? Not trivial! (see Daubechies, 1992 for more details)

Can we take a step further and generate wavelet bases or even *orthonormal* wavelet bases?

Example. Haar basis (1910): $a_0 = 2, b_0 = 1$

$$\psi(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k), \quad j,\ k \in Z$$

A general framework for generating wavelet bases is a **Multiresolution Analisys (MRA)** (Mallat, 1989).

Multiresolution Analsys (MRA)

MRA is a sequence of successive closed approximation spaces $V_j, j \in Z$ in $L_2(R)$ such that

1. $... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$

2. $\overline{\bigcup_j V_j} = L_2(R)$

3. $\bigcap_j V_j = \{0\}$

4. $f(t) \in V_j \Leftrightarrow f(t-k) \in V_j$

5. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$

6. there exists a (*scaling*) function $\phi_0 \in V_0$ such that $\{\phi_{0k}, k \in Z\}$ is an (orthonormal) basis in V_0

Consequence: $V_{j+1} = V_j \oplus W_j$, where W_j is an orthogonal complement of V_j in V_{j+1}

1.
$$\begin{aligned} V_{j+1} &= V_j \oplus W_j = V_{j-1} \oplus W_{j-1} + \oplus W_j = ... \\ &= V_{j_0} \oplus \bigoplus_{l=j_0}^{j} W_l \end{aligned}$$

2. $\overline{\bigoplus_{j \in Z} W_j} = L^2(R)$

3. $f(t) \in W_j \Leftrightarrow f(t-k) \in W_j$

4. $f(t) \in W_j \Leftrightarrow f(2t) \in W_{j+1}$

5. there exists a (*mother wavelet*) function $\psi(t)$ such that $\{\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k),\ k \in Z\}$ forms an orthonormal basis in W_j

$$P_{j+1}f = P_j f + \sum_{k \in Z} (f, \psi_{jk}) \psi_{jk},$$

where $P_j f$ is the projection of f on V_j.

- a series of finer and finer approximations
- switching from the resolution j to $j+1$ adds details to the approximation

$$\begin{aligned} f(t) &= \sum_{j \in Z} \sum_{k \in Z} d_{jk} \psi_{jk}(t) \\ &= \sum_{k \in Z} c_{j_0 k} \phi_{j_0 k}(t) + \sum_{j \geq j_0} \sum_{k \in Z} d_{jk} \psi_{jk}(t), \end{aligned}$$

where

$$\phi_{jk}(t) = 2^{j/2}(2^j t - k), \; \psi_{jk}(t) = 2^{j/2}(2^j t - k)$$

$$d_{jk} = (f, \psi_{jk}), \; c_{jk} = (f, \phi_{jk})$$

MRA allows one to construct various wavelet bases (see Mallat, Daubechies for details)

Remark: the requirement of orthogonality can be relaxed

Choosing a mother wavelet

- the length of support in time and Fourier domains

- the number of vanishing moments (m)

- regularity

- . . .

Examples of Wavelet Bases

- Haar basis

$$supp(\psi) = [0, 1],\ m = 1$$

- Daubechies' compactly supported wavelets $D_N, N \geq 2$

$$supp(\psi) = [-N + 1, N],\ m = N$$

- Symmlets (Daubechies)

- Coiflets

- spline wavelets

- Shannon wavelets, Meyer wavelets (have a compact support in Fourier domain)

- much-much more

From the real axis to the interval

$f(t) \in L^2[0,1]$

- function's extension (periodic, symmetric, antisymmetric)

- boundary wavelets (Meyer, 1992; Cohen *et al.*, 1993)

$$f(t) = \sum_{k=0}^{2^{j_0}-1} c_{j_0 k} \phi_{j_0 k}(t) + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} d_{jk} \psi_{jk}(t)$$

Two-dimensional wavelet bases

tensor-product wavelets

$$\Psi_{j_1,k_1;j_2,k_2}(t_1,t_2) = \psi_{j_1,k_1}(t_1)\psi_{j_2,k_2}(t_2)$$

- $\{\Psi_{j_1,k_1;j_2,k_2}; j_1, k_1, j_2, k_2 \in Z\}$ are indeed wavelets

- form an orthonormal basis in $L^2(R^2)$

two-dimensional MRA

$$\Phi_{j;k_1,k_2}(t_1,t_2) = \phi_{jk_1}(t_1)\phi_{jk_2}(t_2)$$

(rough approximation)

$$\Psi^1_{j;k_1,k_2}(t_1,t_2) = \phi_{jk_1}(t_1)\psi_{jk_2}(t_2)$$

(horizontal direction)

$$\Psi^2_{j;k_1,k_2}(t_1,t_2) = \psi_{jk_1}(t_1)\phi_{jk_2}(t_2)$$

(vertical direction)

$$\Psi^3_{j;k_1,k_2}(t_1,t_2) = \psi_{jk_1}(t_1)\psi_{jk_2}(t_2)$$

(diagonal direction)

$$\Psi^l_{j;k_1,k_2}(t_1,t_2) = 2^j\Psi^l(2^jt_1 - k_1; 2^jt_2 - k_2)$$

$$l = 1,2,3$$

$$
\begin{aligned}
f(t_1,t_2) &= \sum_{k_1,k_2} c_{j_0;k_1,k_2}\Phi_{j_0;k_1,k_2}(t_1,t_2) \\
&+ \sum_{l}\sum_{j\geq j_0}\sum_{k_1,k_2} d^l_{j;k_1,k_2}\Psi^l_{j_0;k_1,k_2}(t_1,t_2),
\end{aligned}
$$

where

$$
c_{j_0;k_1,k_2} = \int\int \Phi_{j;k_1,k_2}(t_1,t_2)f(t_1,t_2)dt_1dt_2
$$

$$
d^l_{j_0;k_1,k_2} = \int\int \Psi^l_{j;k_1,k_2}(t_1,t_2)f(t_1,t_2)dt_1dt_2
$$

$$
l = 1,2,3; \quad j \geq j_0; \quad k_1,\ k_2 \in Z
$$