Compactly Supported Tight Affine Frames with Integer Dilations and Maximum Vanishing Moments[1)]

by

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Abstract. When a Cardinal B-spline of order greater than 1 is used as the scaling function to generate a multiresolution approximation of $L^2 = L^2(\mathbb{R})$ with dilation integer factor $M \geq 2$, the standard "matrix extension" approach for constructing compactly supported tight frames has the limitation that at least one of the tight frame generators does not annihilate any polynomial except the constant. The notion of vanishing moment recovery (VMR) was introduced in our earlier work (and independently by Daubechies, Han, Ron, and Shen) for dilation $M = 2$ to increase the order of vanishing moments. This present paper extends the tight frame results in the above mentioned papers from dilation $M = 2$ to arbitrary integer $M \geq 2$ for any compactly supported M-dilation scaling functions. It is shown, in particular, that M compactly supported tight frame generators suffice, but not $M - 1$ in general. A complete characterization of the M-dilation polynomial symbol is derived for the existence of $M - 1$ such frame generators. Linear spline examples are given for $M = 3, 4$ to demonstrate our constructive approach.

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1. Introduction

While it is well-known that spline functions provide very powerful tools for data representation/manipulation and curve/surface design that are flexible, robust, and computational efficient, there has been a lack of analysis methods in the spline tool-box for quality assessment of the spline representations. In particular, for spline curve/surface editing, only ad hoc procedures that depend on control point manipulation, knot insertion/removal, etc. have been available before the recent introduction of the wavelet approach, particularly in the exciting development of multiresolution or multi-level (MRA) techniques (see [13, 23, 26, 27]).

Spline functions are linear combinations of certain building blocks, called normalized B-splines. For equally-spaced knots, say at the set $\mathbf{Z}$ of all integers, the normalized B-splines with specific smoothness order, say C^{L-2} where $L \geq 1$, are simply integer translates of a single compactly supported non-negative function ϕ_L, called the L-th order Cardinal B-spline, defined by L-fold convolution of the characteristic function of the unit interval. An amazing property of the L-th order Cardinal B-spline ϕ_L is that it provides the multi-level structure for dilation (or scaling) by any integer factor, say $M \geq 2$, via a simple "scaling relation"

$$\phi_L(x) = \sum_{k=0}^{(M-1)L} p_k \phi_L(Mx - k) \tag{1.1}$$

or equivalently, in the Fourier domain,

$$\hat{\phi}_L(\omega) = P_L(z)\hat{\phi}_L(\frac{\omega}{M}), \quad z = e^{-i\omega/M}, \tag{1.2}$$

where

$$P_L(z) := \frac{1}{M} \sum_{k=0}^{(M-1)L} p_k z^k \tag{1.3}$$

is called the (M-dilation) polynomial symbol of the (M-dilation) scaling sequence $\{p_k\}$, given explicitly by

$$P_L(z) = \left(\frac{1 + z + \ldots + z^{M-1}}{M} \right)^L . \tag{1.4}$$

As in the special case $M = 2$ (see e.g. [4]), it is easy to construct $M - 1$ semi-orthogonal wavelets that have compact support and are expressed as linear combinations of $\phi_L(Mx - k)$, $k \in \mathbf{Z}$. However, for $L \geq 2$, the support of the corresponding duals of these wavelets is necessarily all of $(-\infty, \infty)$ (see e.g. [28, 29]). This undesirable property of the "analysis wavelets" makes it difficult to develop very efficient spline tools for spline curve/surface quality assessment.

For $L \geq 2$, when the (total) positivity of ϕ_L does not allow orthogonal (integer) translates, compactly supported tight frames, again expressed as linear combinations of $\phi_L(Mx-k)$, $k \in \mathbf{Z}$, are probably the best alternatives (to replace orthonormal wavelets). However, in place of orthogonality, the important feature of **vanishing moments** must be retained. (Recall that both orthonormal wavelets and semi-orthogonal wavelets, together with their duals, have the L-th order vanishing moments, meaning that they annihilate all polynomials of order L (or degree of $L-1$), when the MRA generated by ϕ_L is considered.) When the (orthogonal) "matrix extension" approach (see [20, 25, 5]) is used to construct Laurent polynomial symbols $Q_1(z), \ldots, Q_N(z)$ of the tight frame (generators) $\psi_1, \ldots, \psi_N$ in terms of the M-dilation symbol $P_L(z)$; more precisely, when the M (orthogonal) equations:

$$P_L(z)\overline{P_L(\omega_M^m z)} + \sum_{n=1}^{N} Q_n(z)\overline{Q_n(\omega_M^m z)} = \delta_{m,0}, \tag{1.5}$$

$m = 0, \ldots, M-1$ with $\omega_M := e^{-i2\pi/M}$, are solved for $Q_1(z), \ldots, Q_N(z)$, to yield $\psi_1, \ldots, \psi_N$ via

$$\hat{\psi}_n(\omega) = Q_n(z)\hat{\phi}_L(\frac{\omega}{M}), \quad z = e^{-i\omega/M}, \tag{1.6}$$

$n = 1, \ldots, N$, it is noted that independent of the choice of N, at least one of the wavelets $\psi_1, \ldots, \psi_N$ has only one vanishing moment. The reason for this is that in (1.5) for $m = 0$, the polynomial

$$\sum_{n=1}^{N} |Q_n(z)|^2 = 1 - |P_L(z)|^2 = 1 - \left| \frac{1 + z + \ldots + z^{M-1}}{M} \right|^{2L}$$

is divisible by $|1-z|^2$, but not by $|1-z|^{2\ell}$ for $\ell > 1$ on $|z| = 1$. Hence, for $M = 2$, the notion of vanishing moment recovery (VMR) functions was introduced in our earlier work [6] (and independently by Daubechies, Han, Ron, and Shen [12], where VMR functions are called fundamental functions), to increase the order of vanishing moments for each of $\psi_1, \ldots, \psi_N$.

In the following discussion, since we are interested in a more general setting (rather than the particular case of Cardinal B-spline ϕ_L), we will replace $P_L(z)$ by a more general Laurent polynomial

$$\begin{aligned} P(z) &= \left(\frac{1 + z + \ldots + z^{M-1}}{M} \right)^L P_0(z) \\ &= \frac{1}{M} \sum_{k=N_1}^{N_2} p_k z^k, \qquad L \geq 1, \text{ and } \quad p_{N_1} p_{N_2} \neq 0, \end{aligned} \tag{1.7}$$

where $P_0(z)$ is some Laurent polynomial not divisible by $(1+z+\ldots+z^{M-1})$, and ϕ_L replaced by an M-dilation "scaling" function ϕ governed by the (M-dilation) scaling relation

$$\phi(x) = \sum_{k=N_1}^{N_2} p_k \phi(Mx - k), \tag{1.8a}$$

or equivalently,

$$\hat{\phi}(\omega) = P(z)\hat{\phi}(\frac{\omega}{M}), \quad z = e^{-i\omega/M}. \tag{1.8b}$$

Here and throughout, the degree of the Laurent polynomial $P(z)$ in (1.7) is defined to be $N_2 - N_1$. As a scaling function, ϕ is assumed to satisfy the Riesz (or stability) condition as well as the low-pass filter condition

$$\hat{\phi}(0) = 1. \tag{1.8c}$$

It is clear that ϕ is compactly supported and that the spaces

$$V_j = \text{clos}_{L^2}\langle \phi(M^j \cdot -k) : \quad k \in \mathbf{Z}\rangle, \qquad j \in \mathbf{Z}, \tag{1.8d}$$

possess the density and nested properties:

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \to L^2, \tag{1.8e}$$

commonly called a multiresolution approximation (MRA) of $L^2 = L^2(\mathbb{R})$. Throughout this paper, ϕ will denote a real-valued compactly supported scaling function that satisfies (1.8a)–(1.8e) as well as the Riesz (or stability) condition. The M (orthogonal) equations, such as (1.5) for $P_L(z)$, are now extended to the M equations

$$S(z^M)P(z)\overline{P(\omega_M^m z)} + \sum_{n=1}^{N} Q_n(z)\overline{Q_n(\omega_M^m z)} = \delta_{m,0}S(z), \ |z| = 1, \tag{1.9}$$

$m = 0, \ldots, M-1$, where $\omega_M := e^{-i2\pi/M}$ and $S(z)$ is a Laurent polynomial that satisfies

$$S(1) = 1. \tag{1.10}$$

The polynomial $S(z)$ in (1.9) will be called a VMR function as coined in [6] for $M = 2$. Clearly, (1.9) is an extension of the "matrix extension" approach, namely (1.5) (where Cardinal B-spline was considered), by extending $S \equiv 1$ to a more general Laurent polynomial that satisfies (1.10), to increase the power ℓ of the factor $(1-z)^\ell$ of each $Q_n(z), n = 1, \ldots, N$; or equivalently, to increase the order of vanishing moments of $\psi_1, \ldots, \psi_N$.

For convenience, the standard notation

$$f_{j,k}(x) := M^{j/2}f(M^j x - k), \quad j, k \in \mathbf{Z}, \tag{1.11}$$

for dilation, translations, and L^2-normalization, will be used. With this notation, a family

$$\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2 \tag{1.12}$$

is said to generate a tight (affine) frame

$$\mathcal{F} := \{\psi_{n;j,k} : 1 \le n \le N,\ j,k \in \mathbf{Z}\} \tag{1.13}$$

of L^2, or for simplicity, we also say that Ψ is a tight (affine) frame, if there exists some constant $A > 0$ (called frame bound constant), such that

$$\sum_{n=1}^{N} \sum_{j,k\in\mathbf{Z}} |\langle f, \psi_{n;j,k}\rangle|^2 = A\|f\|^2, \quad f \in L^2. \tag{1.14}$$

The tight frame $\mathcal{F}$ (or tight frame generators Ψ) is said to be normalized, if ψ_n is replaced by $A^{-1/2}\psi_n$, so that the frame bound constant A in (1.14) is replaced by 1.

The objective of this paper is to extend our work [6] from dilation $M = 2$ to arbitrary $M \ge 2$ on compactly supported affine frames associated with a compactly supported (M-dilation) scaling function and possessing the maximum order of vanishing moments. Although it should not be difficult to generalize the results in this present paper to sibling frames, or more generally to bi-frames, we will not do so. More precisely, we will study the VMR functions in some detail, give a constructive proof of the existence of $N = M$ compactly supported tight affine frames (or frame generators) with maximum order of vanishing moments via VMR Laurent polynomials, characterize the existence of $N = M - 1$ compactly supported tight affine frame generators in terms of certain algebraic structure of the M-dilation Laurent symbol, and demonstrate our theory by considering the construction of compactly supported L-th order Cardinal spline tight frames with L vanishing moments.

This paper is organized as follows. Preliminary results on scaling functions, tight affine frames, transfer operators, and shift-invariant spaces are compiled in Section 2. In Section 3, we derive a characterization of a function $S(z)$ in (1.9)–(1.10) so that Ψ, as defined by $\hat{\psi}_n(\omega) = Q_n(z)\hat{\phi}(\omega/M)$, $z = e^{-i\omega/M}$, generates a tight frame $\mathcal{F}$ of L^2. The characterization, as stated in Theorem 3.1, is simply $S(z) \ge 0$ for $|z| = 1$. This function $S(z)$ is used to generate additional vanishing moments for the tight frames, and hence is called a *vanishing moments recovery* (VMR) function. Its intrinsic property with the scaling symbols is formulated and studied in Section 4. In particular, a necessary and sufficient condition for $S(z)$ to be a VMR function in the construction of a tight frame associated with the scaling function of an MRA is established (see Theorem 4.1). As an application of Theorem 4.1, we show, in Section 5, that a tight affine frame $\mathcal{F}$ with M generators associated with an M-dilation scaling function ϕ of an MRA, having maximum vanishing moments as governed by ϕ, can always be constructed (see Theorem 5.1). Of course, if the MRA is generated by a compactly supported orthonormal M-dilation scaling function, then there exist $M - 1$ compactly supported orthonormal wavelets (and hence a tight affine frame), that have maximum vanishing moments. In Section 6, we give a necessary and sufficient condition for the existence of tight affine frames with $M-1$ compactly supported generators in V_1 (see Theorem 6.1), when the compactly supported scaling function is not orthonormal. Examples of linear-spline compactly supported tight frames with dilation 3 and 4, and having maximum (that is, second order) vanishing moments, are presented in the last section.

2. Preliminaries

In this section, we introduce the necessary notations and compile certain preliminary results on tight frames, transfer operators, and shift-invariant spaces to facilitate our later development.

2.1. Scaling Functions and Tight Frames

Throughout this paper we only consider compactly supported scaling functions ϕ with finite and real-valued M-dilation scaling sequences $\{p_k\}$, so that (1.7) and (1.8a)–(1.8e) are valid. The Fourier transform, defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx$$

for integrable functions f, is extended to tempered distributions. Hence, the Fourier transform of a compactly supported distribution is an entire (or analytic) function on the complex plane.

The objective of this paper is to study tight frames of L^2 generated by a family $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2$, that are defined by scaling relations

$$\hat{\psi}_n(\omega) = Q_n(z)\hat{\phi}(\omega/M), \quad z = e^{-i\omega/M}, \quad n = 1, \ldots, N, \tag{2.1}$$

where $Q_1, \ldots, Q_N$ are Laurent polynomials that have real coefficients and vanish at $z = 1$. In particular, we are interested in the formulation

$$Q_n(z) = (1-z)^{L_n} q_n(z), \qquad q_n(1) \neq 0, \tag{2.2}$$

where $L_n \geq 1$. In other words, L_n is the order of vanishing moments of ψ_n. The largest possible L_n is L. We remark that under the assumption $L_n \geq 1$, ψ_n already satisfy the Bessel (i.e. the upper frame bound) condition, as shown in [7].

The following result, which is a direct generalization of a result in [9, 18] for the case $M = 2$, will be used in our later development.

Lemma 2.1. *Let ϕ be a compactly supported L^2 function that satisfies* (1.8a), (1.8c), *and the Riesz condition. Then there do not exist Laurent polynomials $R_1(z)$ and $R_2(z)$ such that R_1 is not a monomial, all the non-zero roots of $R_1(z)$ are on $|z| = 1$, $R_1(1) \neq 0$ and*

$$P(z) = R_1(z^M)R_2(z)/R_1(z). \tag{2.3}$$

Proof. Suppose that the Laurent polynomial symbol $P(z)$ of ϕ has the form of (2.3) for some Laurent polynomials $R_1(z)$ and $R_2(z)$ such that R_1 is not a monomial, all the non-zero roots of $R_1(z)$ are on $|z| = 1$, and $R_1(1) \neq 0$. It follows from (1.8c) and (2.3) that

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} P(e^{-iM^{-j}\omega}) = \prod_{j=1}^{\infty} R_2(e^{-iM^{-j}\omega}) \times \frac{R_1(e^{-i\omega})}{R_1(1)}. \tag{2.4}$$

By hypothesis, $R_1(z)$ has at least one root on $|z| = 1$, say $e^{-i\omega_0}$, $\omega_0 \in \mathbb{R}$. Then it follows from (2.4) that $\hat{\phi}(\omega_0 + 2k\pi) = 0$ for all $k \in \mathbf{Z}$, so that the continuous function (actually trigonometric polynomial) $\sum_k |\hat{\phi}(\omega + 2\pi k)|^2$ has a zero at $\omega = \omega_0$, or ϕ cannot satisfy the Riesz condition (see [4]). ■

The following result gives a precise characterization of tight affine frames of L^2. Note that no reference is made to an underlying scaling function.

Proposition 2.2. *The affine family $\mathcal{F}$ in* (1.13) *is a tight frame of L^2 if and only if $\Psi = \{\psi_1, \ldots, \psi_N\}$ satisfies*

$$\sum_{j \in \mathbf{Z}} \sum_{n=1}^{N} |\hat{\psi}_n(M^j\omega)|^2 = A, \tag{2.5}$$

and

$$\sum_{j=0}^{\infty} \sum_{n=1}^{N} \hat{\psi}_n(M^j\omega)\overline{\widehat{\psi_n}(M^j(\omega + 2k\pi))} = 0, \qquad k \notin M\mathbf{Z}, \tag{2.6}$$

a.e. in $\mathbb{R}$, where A is a positive constant. Furthermore, if Ψ satisfies (2.5) *and* (2.6), *then the constant A is the (tight) frame bound of $\mathcal{F}$.*

The above result is well documented in the book [17] for orthonormal wavelets with $M = 2$. It was later generalized to tight frames with integer dilation (see [14, 25]). A complete extension to tight affine frames with arbitrary real dilation was recently established in [8].

2.2. Transfer Operators

The notion of transfer operator has appeared in many investigations. In this paper, we use the eigenfunctions of a transfer operator to establish the existence of a compactly supported tight affine frame with M generators associated with an MRA. As usual, the space of 2π-periodic functions with ℓ_1-Fourier coefficients is called the Wiener class and denoted by $\mathcal{W}$. Given a Laurent polynomial H, we define the corresponding *transfer operator* $T_{|H|^2}$ on $\mathcal{W}$ by

$$T_{|H|^2} f(z^M) = \sum_{m=0}^{M-1} |H(e^{-i2m\pi/M} z)|^2 f(e^{-i2m\pi/M} z). \tag{2.7}$$

The following result was given in [19, Theorem 2.3].

Lemma 2.3. *Let ϕ be a compactly supported scaling function with dilation factor M, and Laurent polynomial $P(z)$. Then 1 is a simple eigenvalue of the transfer operator $T_{|P|^2}$ on $\mathcal{W}$, and all of the other eigenvalues of $T_{|P|^2}$ lie inside the unit circle. Moreover, the Laurent polynomial*

$$\Phi(z) := \sum_{k\in\mathbf{Z}} A_\phi(k) z^k = \sum_{k\in\mathbf{Z}} |\hat{\phi}(\omega + 2k\pi)|^2, \quad z = e^{-i\omega}, \tag{2.8}$$

is an eigenfunction of the transfer operator $T_{|P|^2}$ with eigenvalue 1, where A_ϕ is the auto-correlation of ϕ defined by

$$A_\phi(x) := \int_{\mathbb{R}} \phi(t)\overline{\phi(t+x)}dt. \tag{2.9}$$

The formulation in (2.8) follows from Poisson Summation formula. In the following, we use the notation

$$\Pi_{\mathcal{N}} := \left\{ \sum_{j=-\mathcal{N}}^{\mathcal{N}} f_j z^j \ : \ f_j \in \mathbb{R} \right\}.$$

Note that each Laurent polynomial in $\Pi_{\mathcal{N}}$ has degree at most $2\mathcal{N}$. It is clear that for sufficiently large $\mathcal{N}$, $\Pi_{\mathcal{N}}$ is an invariant subspace of the transfer operator $T_{|H|^2}$. In particular, if $|H|^2 \in \Pi_{\mathcal{N}_0}$, then we can choose any integer $\mathcal{N} \geq \mathcal{N}_0/(M-1)$ to define an invariant subspace $\Pi_{\mathcal{N}}$. For $L \geq 0$, we set

$$E_{\mathcal{N},L} = \{0 \leq p \in \Pi_{\mathcal{N}} : \quad p(z) = O((1-z)^L) \quad \text{near } z = 1\}.$$

It is easy to see that if H is a Laurent polynomial of degree $\mathcal{N}_0$ and has the form

$$H(z) = \left(\frac{1 + z + \ldots + z^{M-1}}{M}\right)^L H_0(z), \tag{2.10}$$

where L is a nonnegative integer for some Laurent polynomial H_0, then $E_{\mathcal{N},2L}$ is invariant under the transfer operator $T_{|H|^2}$ for any integer $\mathcal{N} \geq 2\mathcal{N}_0/(M-1)$. Moreover, for any Laurent polynomial $f \in \Pi_{\mathcal{N}}$, we have

$$T_{|H|^2} f_L(z^M) = M^{-2L} \left| \frac{1-z^M}{2} \right|^{2L} (T_{|H_0|^2} f)(z^M), \tag{2.11}$$

where $f_L(z) = \left|\frac{1-z}{2}\right|^{2L} f(z)$. Therefore, as a consequence of (2.11) and Lemma 2.3, we have the following estimate for the spectral radius of the transfer operator associated with a scaling function.

Lemma 2.4. *Let $\phi \in L^2$ be a compactly supported scaling function with dilation factor M, and M-dilation symbol $P(z)$ satisfying (1.7) for some Laurent polynomial P_0 of degree $\mathcal{N}_0$, where $\mathcal{N}_0 \geq 0$. Then the spectral radius of $T_{|P_0|^2}$ on $\Pi_{\mathcal{N}}$ for $\mathcal{N} \geq 2\mathcal{N}_0/(M-1)$ is strictly less than M^{2L}.*

Next we analyze the *irreducibility* of the transfer operator $T_{|H|^2}$. According to [24], irreducibility of the transfer operator T is defined by the following property: if $Tf \leq \alpha f$ for some positive constant α and $f \geq 0$, then $f > 0$ unless $f \equiv 0$.

Lemma 2.5. *Let H be a Laurent polynomial of degree $\mathcal{N}_0$. Assume that there does not exist a nonzero Laurent polynomial H_1 which is not a monomial and with all roots on $|z| = 1$ such that $H(z)H_1(z)/H_1(z^M)$ remains to be a Laurent polynomial. Then $T_{|H|^2}$ is irreducible on $\Pi_{\mathcal{N}}$ for $\mathcal{N} \geq 2\mathcal{N}_0/(M-1)$.*

The argument for the proof of the above result is similar to the derivation of the result for using the symbol to characterize the stability of a scaling function ([9, 18, 30]). We remark that a factor of the form $\frac{1}{M}(1+z+\ldots+z^{M-1})$ in (2.10) cannot be present in H, if the assumption in the lemma is satisfied. For completeness, we include a proof of the lemma here.

Proof. We will use the trigonometric polynomial expression for the Laurent polynomials, and abuse the notation by using $H(\omega)$ for $H(e^{-i\omega})$, etc. Let $0 \leq f \in \Pi_{\mathcal{N}}$ and $Tf \leq \alpha f$. If $f > 0$, there is nothing to be shown. Let us assume that there exists $\omega_0 \in (-\pi, \pi]$ such that $f(\omega_0) = 0$. Then it follows from $Tf \leq \alpha f$ and $f \geq 0$ that

$$\left| H\Big(\frac{\omega_0 + 2m\pi}{M}\Big)\right|^2 f\Big(\frac{\omega_0 + 2m\pi}{M}\Big) = 0, \qquad m = 0, \ldots, M-1.$$

By the assumption on H, there exists an integer $0 \leq m(\omega_0) \leq M-1$ such that $H((\omega_0 + 2m(\omega_0)\pi)/M) \neq 0$; otherwise, H, by itself, would be divisible by $H_1(M\omega)$, where $H_1(\omega) = e^{i\omega} - e^{i\omega_0}$. This leads to $f((\omega_0 + 2m(\omega_0)\pi)/M) = 0$. Hence, there exists $\omega_1 \in (-\pi, \pi]$ such that $f(\omega_1) = 0$ and $M\omega_1 \equiv \omega_0 \pmod{2\pi}$. By following this procedure iteratively, we derive a sequence $\{\omega_k : k \geq 0\} \subset (-\pi, \pi]$, such that $f(\omega_k) = 0$ and

$$M\omega_{k+1} \equiv \omega_k \pmod{2\pi} \quad \text{for all} \quad k \geq 0. \tag{2.12}$$

Recall that f has finitely many roots on $(-\pi, \pi]$. Hence, $\omega_{k_1} = \omega_0$ for some $k_1 \geq 1$. For simplicity, we denote the smallest positive integer k such that $\omega_k = \omega_0$ by k_1. We have

$$H\Big(\omega_k + \frac{2m\pi}{M}\Big) = 0 \tag{2.13}$$

for all $1 \leq m \leq M-1$ and $0 \leq k \leq k_1 - 1$. (The proof of (2.13) is similar to the one in [10, pp. 192] where $M = 2$.)

On one hand, it follows from (2.12) and the identity $(e^{-iM\omega} - e^{-i\alpha}) = \prod_{m=0}^{M-1}(e^{-i\omega} - e^{-i((\alpha+2m\pi)/M)})$ that

$$\prod_{k=0}^{k_1-1}\prod_{m=0}^{M-1}(e^{-i\omega} - e^{-i(\omega_k+2m\pi/M)}) = \prod_{k=0}^{k_1-1}(e^{-iM\omega} - e^{-i\omega_k}). \tag{2.14}$$

Hence, by (2.13), $H(\omega)\prod_{k=0}^{k_1-1}(e^{-i\omega} - e^{-i\omega_k})$ is divisible by the trigonometric polynomial on the left-hand side of (2.14). We thus obtain $H(\omega) = H_1(M\omega)H_2(\omega)/H_1(\omega)$, where $H_1(\omega) = \prod_{k=0}^{k_1-1}(e^{-i\omega} - e^{-i\omega_k})$ and $H_2(\omega)$ is a trigonometric polynomial. This contradicts with the assumptions on H. ∎

Based on the Perron-Frobenius theory (see [24]) and the irreducibility of the transfer operator given in Lemma 2.5, we have the following result concerning $T_{|H|^2}$.

Lemma 2.6. *Let H be a Laurent polynomial of degree $\mathcal{N}_0$, and ρ be the spectral radius of the transfer operator $T_{|H|^2}$ on $\Pi_{\mathcal{N}}$, where $\mathcal{N} \geq 2\mathcal{N}_0/(M-1)$. Assume that there does not exist a nonzero Laurent polynomial H_1, which is not a monomial, such that $H(z)H_1(z)/H_1(z^M)$ remains to be a Laurent polynomial. Then there exists $F \in \Pi_{\mathcal{N}}$ such that $T_{|H|^2}F = \rho F$ and $F(z) > 0$ on $|z| = 1$.*

2.3. Frames in Shift-Invariant Spaces

To study tight affine frames with $M-1$ generators associated with an M-dilation scaling function ϕ, the topic to be discussed in Section 6, it is convenient to investigate certain frame property for shift-invariant spaces. For a finite collection $\Psi = \{\psi_1, \ldots, \psi_N\}$ of L^2 functions, we define the corresponding shift-invariant space $W(\Psi) = W(\psi_1, \ldots, \psi_N)$ as (infinite) linear combinations of integer shifts in terms of ℓ^2 sequences, namely:

$$W(\Psi) = \left\{\sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} d_{nk}\psi_n(\cdot - k) : \{d_{nk}\}_{k\in\mathbf{Z}} \in \ell^2, 1 \leq n \leq N\right\}.$$

We will say that Ψ is a frame for the shift-invariant space $W(\Psi)$ if there exist positive constants $A \le B$, called frame bounds, such that

$$A\|f\|_2^2 \le \sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} |\langle f, \psi_n(\cdot - k)\rangle|^2 \le B\|f\|_2^2, \qquad f \in W(\Psi).$$

Using the characterization of frames for shift-invariant spaces in [1, 2, 3], we have the following result for the functions chosen from a multiresolution approximation.

Proposition 2.7. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by a compactly supported M-dilation scaling function ϕ. Let $\psi_1, \ldots, \psi_N \in V_1$ be compactly supported L^2 functions defined by $\hat{\psi}_n(\omega) = Q_n(z)\hat{\phi}(\omega/M)$ for some Laurent polynomials $Q_n(z), 1 \le n \le N, z = e^{-i\omega/M}$. Then $\{\psi_1, \ldots, \psi_N\}$ is a frame of the shift-invariant space $W(\psi_1, \ldots, \psi_N)$ if and only if the rank of the $N \times M$ matrix $\left[Q_n(e^{-i2m\pi/M}z)\right]_{1\le n\le N, 0\le m\le M-1}$ is independent of z on $|z| = 1$.*

3. M-Dilation Tight Affine Frames Associated with MRA

In this section, we give another characterization of tight affine frames associated with some M-dilation scaling function. This formulation will be more convenient for explicit construction of tight affine frames associated with some MRA. We will employ the notation ℓ_0 for the space of all finite sequences.

Theorem 3.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by a compactly supported M-dilation scaling function ϕ. Consider*

$$\psi_n = \sum_{j\in\mathbf{Z}} q_{nj}\phi(M \cdot - j), \quad 1 \le n \le N, \tag{3.1}$$

in V_1 defined by some $\{q_{nj}\}_{j\in\mathbf{Z}} \in \ell_0, 1 \le n \le N$. Then $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a normalized (i.e. $A = 1$ in (1.14) or (2.5)) tight affine frame if and only if there exists a Laurent polynomial $S(z)$ that satisfies

(i) $S(1) = 1$;
(ii) $S(z) \ge 0$ *on* $|z| = 1$; *and*
(iii) *for all* $m = 0, \ldots, M-1$,

$$S(z^M)P(z)P(e^{i2m\pi/M}z^{-1}) + \sum_{n=1}^{N} Q_n(z)Q_n(e^{i2m\pi/M}z^{-1}) = \delta_{m0}S(z), \tag{3.2}$$

where $Q_n(z) = \frac{1}{M}\sum_{j\in\mathbf{Z}} q_{nj}z^j, 1 \le n \le N$.

The above result is a generalization of a result in [6] from $M = 2$ to arbitrary integer $M \ge 2$. In view of the independent development in [6] and [12], we extend the proofs in these two recent papers to arbitrary integer dilations. The sufficiency direction extends and simplifies the proof for bi-frames in [11], and the proof for the necessity direction will be a generalization of the proof for $M = 2$ in [6].

Proof of Theorem 3.1.
(i) *Sufficiency.* Let $\tilde{\phi}$ be defined by $\widehat{\tilde{\phi}}(\omega) = S(e^{-i\omega})\hat{\phi}(\omega)$. For $j \in \mathbf{Z}$, we consider

$$\Lambda_j f := \sum_{k\in\mathbf{Z}} \langle f, \phi_{j,k}\rangle \tilde{\phi}_{j,k}, \qquad f \in L^2,$$

and

$$\Omega_j f := \sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} \langle f, \psi_{n;j,k}\rangle \psi_{n;j,k}.$$

Then by direct computations and using the notation $z = e^{-i\omega/M}$, we obtain

$$\begin{aligned}
\widehat{\Omega_0 f}(\omega) =&(2\pi)^{-1}\sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} \widehat{\psi}_n(\omega)e^{-ik\omega}\langle \hat{f}, \widehat{\psi}_n(\cdot)e^{-ik\cdot}\rangle \\
=&\sum_{n=1}^{N}\sum_{l\in\mathbf{Z}} \hat{f}(\omega+2l\pi)\overline{\widehat{\psi}_n(\omega+2l\pi)}\widehat{\psi}_n(\omega) \\
=&\sum_{l'\in\mathbf{Z}}\sum_{m=0}^{M-1} \hat{f}(\omega+2Ml'\pi+2m\pi)\overline{\widehat{\phi}\Big(\frac{\omega+2m\pi}{M}+2l'\pi\Big)}\widehat{\phi}\Big(\frac{\omega}{M}\Big) \\
&\times\Big(\sum_{n=1}^{N} Q_n(e^{i2m\pi/M}z^{-1})Q_n(z)\Big) \\
=&\sum_{l'\in\mathbf{Z}}\sum_{m=0}^{M-1} \hat{f}(\omega+2Ml'\pi+2m\pi)\overline{\widehat{\phi}\Big(\frac{\omega+2m\pi}{M}+2l'\pi\Big)}\widehat{\phi}\Big(\frac{\omega}{M}\Big) \\
&\times\Big(-S(z^M)P(e^{i2m\pi/M}z^{-1})P(z)+\delta_{m0}S(z)\Big) \\
=&-S(z^M)\sum_{l\in\mathbf{Z}} \hat{f}(\omega+2l\pi)\overline{\widehat{\phi}(\omega+2l\pi)}\widehat{\phi}(\omega) \\
&+S(z)\sum_{l\in\mathbf{Z}} \hat{f}(\omega+2lM\pi)\overline{\widehat{\phi}\Big(\frac{\omega}{M}+2l\pi\Big)}\widehat{\phi}\Big(\frac{\omega}{M}\Big) \\
=&-\widehat{\Lambda_0 f}(\omega)+\widehat{\Lambda_1 f}(\omega).
\end{aligned}$$

This proves that $\Omega_0 f = \Lambda_1 f - \Lambda_0 f$ for all $f \in L^2$. Hence, it follows from dilation invariance that

$$\Omega_j f = \Lambda_{j+1} f - \Lambda_j f, \qquad f \in L^2. \tag{3.3}$$

Clearly,

$$\sum_{j\in\mathbf{Z}}\langle\Omega_j f, f\rangle = \sum_{n=1}^{N}\sum_{j,k\in\mathbf{Z}}|\langle f,\psi_{n;j,k}\rangle|^2, \qquad f\in L^2. \tag{3.4}$$

Therefore, in view of (3.3) and (3.4), it suffices to prove that

$$\lim_{j\to\infty}\|\Lambda_j f - f\|_2 = 0 \quad \text{and} \quad \lim_{j\to-\infty}\|\Lambda_j f\|_2 = 0, \quad f\in L^2.$$

This can be easily verified, since $S(1)\neq 0$ and ϕ is a compactly supported L^2 function with $\hat{\phi}(0)=1$ (see, for instance, [22] or [11, Lemma 2.2]).

(ii) *Necessity.* Suppose that Ψ generates a normalized tight affine frame $\mathcal{F}$ of L^2. Then consider an auxiliary function

$$\begin{aligned}\Theta(\omega) &:= \frac{1}{|\hat{\phi}(\omega)|^2}\sum_{j=1}^{\infty}\sum_{n=1}^{N}|\hat{\psi}_n(M^j\omega)|^2\\ &= \sum_{j=1}^{\infty}\sum_{n=1}^{N}|Q_n(z^{M^j})|^2\prod_{k=1}^{j-1}|P(z^{M^k})|^2,\end{aligned} \tag{3.5}$$

which already appeared in [25]. Since $\hat{\phi}$ is an entire function, it has only isolated zeros, if any at all. Hence, Θ is a measurable function, which can be defined by Laurent polynomials in the second line of (3.5). This shows that Θ is 2π-periodic. By (3.5), we can write $\Theta(\omega)$ in the z-notation, to be denoted by $S(z^M)$, namely $S(z^M) := \Theta(\omega), z = e^{-i\omega/M}$. Obviously, $S(z)\geq 0$ on $|z|=1$. We need to show that $S(z)$ is a Laurent polynomial. By (2.6), we have

$$\begin{aligned}0 &= \sum_{j\geq 0}\sum_{n=1}^{N}\hat{\psi}_n(M^j\omega)\overline{\widehat{\psi_n}(M^j(\omega+2k\pi))}\\ &= \hat{\phi}(\omega/M)\overline{\hat{\phi}(\omega/M+2k\pi/M)}\left[\sum_{n=1}^{N}Q_n(z)Q_n(e^{i2k\pi/M}z^{-1}) + P(z)P(e^{i2k\pi/M}z^{-1})\Theta(\omega)\right],\end{aligned}$$

for almost all $\omega\in\mathbb{R}$ and $k\notin M\mathbf{Z}$. Then, the analyticity of $\hat{\phi}$ leads to

$$0 = \sum_{n=1}^{N}Q_n(z)Q_n(e^{i2k\pi/M}z^{-1}) + P(z)P(e^{i2k\pi/M}z^{-1})S(z^M), \qquad \text{a.e. } z\in\mathbb{T}. \tag{3.6}$$

Taking $k=1$ in (3.6), we obtain

$$S(z^M) = -\Big(P(z)P(e^{i2\pi/M}z^{-1})\Big)^{-1}\sum_{n=1}^{N}Q_n(z)Q_n(e^{i2\pi/M}z^{-1}), \tag{3.7}$$

which is a rational Laurent function. Write

$$S(z) = a(z)/b(z), \tag{3.8}$$

where $a(z)$ and $b(z)$ are Laurent polynomials whose only common factor is a monomial. On one hand, (2.5) leads to

$$1 = \sum_{j\in\mathbf{Z}}\sum_{n=1}^{N} |\hat{\psi}_n(M^j\omega)|^2 = \lim_{J\to-\infty}\sum_{j=J}^{\infty}\sum_{n=1}^{N} |\hat{\psi}_n(M^j\omega)|^2 = \lim_{J\to-\infty}\left[\Theta(M^J\omega)|\hat{\phi}(M^{J-1}\omega)|^2\right].$$

Hence, the continuity of $\hat{\phi}$ at $\omega = 0$ implies that

$$\lim_{J\to-\infty}\Theta(M^J\omega) = 1$$

for almost all $\omega \in \mathbb{R}$. It follows that $S(1) = \Theta(0) = 1$, and hence,

$$b(1) \neq 0. \tag{3.9}$$

On the other hand, (3.5) can also be written as

$$S(z) = \sum_{n=1}^{N} |Q_n(z)|^2 + |P(z)|^2 S(z^M). \tag{3.10}$$

So, substituting the expression (3.8) for S in (3.10) gives

$$\frac{a(z^M)}{b(z^M)} \times |P(z)|^2 + \sum_{n=1}^{N} |Q_n(z)|^2 = \frac{a(z)}{b(z)}. \tag{3.11}$$

Therefore, $a(z)b(z^M)/b(z)$ must be a Laurent polynomial. Since the common factor of $a(z)$ and $b(z)$ is a monomial, $b(z^M)/b(z)$ is indeed a Laurent polynomial and hence all non-zero roots of $b(z)$ are on the circle $|z| = 1$. Moving the term $\sum_{n=0}^{N} |Q_n(z)|^2$ to the right-hand side of equation (3.11), and then multiplying by $b(z^M)$ on both sides lead to

$$a(z^M)|P(z)|^2 = R_1(z)b(z^M)/b(z), \tag{3.12}$$

where R_1 is a Laurent polynomial. Again using the assumption that the only common factor of $a(z)$ and $b(z)$ is a monomial, we conclude from (3.8) that

$$|P(z)|^2 = R_2(z)b(z^M)/b(z), \tag{3.13}$$

where R_2 is a Laurent polynomial as well. It is easy to see that $|P(z)|^2$ is the M-dilation symbol of the scaling function A_ϕ, which is the autocorrelation function of ϕ in (2.9) so that $\hat{A}_\phi(\omega) = |\hat{\phi}(\omega)|^2$. Then since $\hat{\phi}$ is continuous, the stability condition of ϕ, that is $\sum_{k\in\mathbf{Z}} |\hat{\phi}(\omega + 2k\pi)|^2 \geq C_1 > 0$ for some C_1, implies that $\sum_{k\in\mathbf{Z}} |\hat{A}_\phi(\omega + 2k\pi)|^2 \geq C_2 > 0$ for some C_2. Thus, A_ϕ also satisfies the Riesz condition. Hence, (3.9) and (3.13) lead to a contradiction of Lemma 2.1 unless $b(z)$ is a monomial. Consequently $S(z)$ is a Laurent polynomial by (3.8). By using $S(z)$ in (3.6) and (3.10), we complete the proof for the necessity direction. ■

4. VMR Functions

The function $S(z)$ in Theorem 3.1 plays an important role in our explicit construction of tight affine frames with additional vanishing moments. In this section, we give a necessary and sufficient condition on the Laurent polynomials $S(z)$ under which the conditions (i)–(iii) hold for some Laurent polynomials $Q_n(z),\ 1 \le n \le N$.

In this section we restrict our discussion to the case where ϕ and ψ_n, $1 \le n \le N$, are real-valued and Q_n, $1 \le n \le N$, are Laurent polynomials with real coefficients. We will make use of the identity $\overline{Q_n(z)} = Q_n(1/z),\ |z| = 1$, whenever convenient.

Theorem 4.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by some compactly supported M-dilation scaling function ϕ and $P(z)$ be its M-dilation symbol. Let S be a Laurent polynomial with real coefficients such that $S(1) = 1$, and $S(z) \ge 0$ on $|z| = 1$. Then there exist Laurent polynomials $Q_n(z), 1 \le n \le N$ satisfying* (3.2), *if and only if*

$$\prod_{m=0}^{M-1} S(e^{-i2m\pi/M}z) - S(z^M) \sum_{m=0}^{M-1} |P(e^{-i2m\pi/M}z)|^2 \prod_{j=0, j\ne m}^{M-1} S(e^{-i2j\pi/M}z) \ge 0 \qquad (4.1)$$

on $|z| = 1$.

The condition (4.1) on $S(z)$ will be used later for the construction of tight affine frames associated with ϕ. The corresponding result for $M = 2$ was established in [6], but the proof given here is a little different.

Recall that an $M \times M$ matrix A is positive semi-definite if $\bar{c}^T Ac \ge 0$ for any vector $c \in \mathbb{C}^M$. To prove Theorem 4.1, we need the following matrix-valued Fejer-Riesz Lemma with real coefficients in [16] (see also the references therein for the matrix-valued Riesz Lemma).

Lemma 4.2. *Let A be an $M \times M$ matrix whose entries are Laurent polynomials with real coefficients, such that $A(z) = A(1/z)^T$ is positive semi-definite on $|z| = 1$. Then there exists an $M \times M$ matrix R, whose entries are Laurent polynomials with real coefficients, such that*

$$A(z) = R(z)R(1/z)^T, \qquad |z| = 1. \qquad (4.2)$$

An algorithm for computing such a factorization of A is given in [16]. Another simple algorithm for the case $M = 2$ was established in [6]. For the proof of Theorem 4.1, we also need the polyphase form of equation (3.2), as follows.

Lemma 4.3. *Let S, P and Q_n, $1 \le n \le N$, be Laurent polynomials with real coefficients, and let $P_m, Q_{n,m}$ and $S_{m,m'}$ be defined by*

$$P(z) = \sum_{m=0}^{M-1} z^m P_m(z^M), \quad Q_n(z) = \sum_{m=0}^{M-1} z^m Q_{n,m}(z^M), \quad 1 \le n \le N, \qquad (4.3)$$

and

$$z^{m'}S(z)=\sum_{m=0}^{M-1} z^m S_{m,m'}(z^M), \quad 0\le m'\le M-1. \tag{4.4}$$

Moreover, set

$$\mathbf{R}(z):=[Q_{n,m}(z)]_{1\le n\le N, 0\le m\le M-1} \tag{4.5}$$

and

$$\tilde{\mathcal{M}}(z):=\Big[\frac{1}{M}S_{m,m'}(z)-S(z)P_m(z)P_{m'}(1/z)\Big]_{0\le m,m'\le M-1}. \tag{4.6}$$

Then $S(z), P(z)$ and $Q_n(z), 1\le n\le N$, satisfy (3.2) if and only if

$$\mathbf{R}(z)^T\mathbf{R}(1/z)=\tilde{\mathcal{M}}(z), \qquad |z|=1. \tag{4.7}$$

Proof. By using the polyphase decomposition (4.3), equation (3.2) becomes

$$\sum_{m'=0}^{M-1}\sum_{m=0}^{M-1}\Big(P_m(z^M)P_{m'}(z^{-M})S(z^M) + \sum_{n=1}^{N} Q_{n,m}(z^M)Q_{n,m'}(z^{-M})\Big)z^{(m-m')}e^{i2sm'\pi/M}=S(z)\delta_{s0},$$

for all $s=0,\ldots,M-1$. Thus,

$$\sum_{m=0}^{M-1}\Big(P_m(z^M)P_{m'}(z^{-M})S(z^M)+ \sum_{n=1}^{N} Q_{n,m}(z^M)Q_{n,m'}(z^{-M})\Big)z^{(m-m')}=\frac{1}{M}S(z) \tag{4.8}$$

for all $m'=0,\ldots,M-1$. This together with (4.4) lead to

$$S(z)P_m(z)P_{m'}(1/z)+\sum_{n=1}^{N} Q_{n,m}(z)Q_{n,m'}(1/z)=\frac{1}{M}S_{m,m'}(z), \tag{4.9}$$

where $0\le m,m'\le M-1$. The assertion then follows. ■

Proof of Theorem 4.1.

(i) *Necessity.* By the assumption, there exist Laurent polynomials $Q_n(z), 1\le n\le N$, such that

$$S(z^M)P(z)P(e^{i2m\pi/M}z^{-1})+\sum_{n=1}^{N} Q_n(z)Q_n(e^{i2m\pi/M}z^{-1})=S(z)\delta_{m0}.$$

Therefore, for any $m, m' = 0, \dots, M-1$, we have

$$\begin{aligned}&\sum_{n=1}^{N} Q_n(e^{-i2m\pi/M} z) Q_n(e^{i2m'\pi/M} z^{-1}) \\ =&\delta_{mm'} S(e^{-i2m\pi/M} z) - S(z^M) P(e^{-i2m\pi/M} z) P(e^{i2m'\pi/M} z^{-1}).\end{aligned} \tag{4.10}$$

This implies that the $M \times M$ matrix

$$\begin{aligned}\mathcal{M}(z) =&\mathrm{diag}\left(S(z), \dots, S(e^{-i2m\pi/M} z)\right) \\ &- S(z^M)\Big[P(e^{-i2m\pi/M} z) P(e^{-i2m'\pi/M} z^{-1})\Big]_{0 \le m, m' \le M-1}\end{aligned} \tag{4.11}$$

is positive semi-definite on $|z| = 1$, and thus,

$$\det \mathcal{M}(z) \ge 0, \qquad |z| = 1. \tag{4.12}$$

For a diagonal matrix $A = \mathrm{diag}(a_1, \dots, a_M)$, and vectors $u = (u_1, \dots, u_M)^T$ and $w = (w_1, \dots, w_M)^T$ in $\mathbb{C}^M$, we have, by induction on $M \ge 1$, that

$$\det(A - u^T w) = \prod_{i=1}^{M} a_i - \sum_{i=1}^{M} u_i w_i \prod_{1 \le j \le M, j \ne i} a_j. \tag{4.13}$$

Note that we may write the matrix $\mathcal{M}(z)$ as $\mathcal{M}(z) = A - u^T w$, where we define $A = \mathrm{diag}(S(z), \dots, S(z\ e^{-i2(M-1)\pi/M}))$, $u = S(z^M)(P(z), \dots, P(e^{-i2(M-1)\pi/M} z))^T \in \mathbb{C}^M$ and $w = (P(z^{-1}), \dots, P(e^{i2(M-1)\pi/M} z^{-1}))^T \in \mathbb{C}^M$. Therefore, by (4.13), we have

$$\begin{aligned}\det \mathcal{M}(z) =& \prod_{m=0}^{M-1} S(e^{-i2m\pi/M} z) \\ &- S(z^M) \sum_{m=0}^{M-1} \Big|P(e^{-i2m\pi/M} z)\Big|^2 \prod_{i=0, i \ne m}^{M-1} S(e^{-i2m\pi/M} z).\end{aligned} \tag{4.14}$$

Thus, the inequality (4.1) follows from (4.12) and (4.14).

(ii) *Sufficiency.* Let $\mathcal{M}(z)$ be defined as in (4.11). We consider the sub-matrices $\mathcal{M}_r(z)$, $1 \le r \le M$, which consist of the upper-left square corner of $\mathcal{M}(z)$ of block size $r \times r$. Note that $\mathcal{M}_r(z)$ takes on a form similar to the right-hand side of (4.11), with the exception that certain truncation of the diagonal matrix and vectors occurs. An argument similar to the proof of the identity (4.14) gives

$$
\begin{aligned}
\det \mathcal{M}_r(z) =& \prod_{m=0}^{r-1} S(e^{-i2m\pi/M}z) \\
& - S(z^M)\sum_{m=0}^{r-1}\Big|P(e^{-i2m\pi/M}z)\Big|^2 \prod_{j=0, j\neq m}^{r-1} S(e^{-i2j\pi/M}z) \\
=& \prod_{m=0}^{r-1} S(e^{-i2m\pi/M}z)S(z^M) \\
& \times \Big(\frac{1}{S(z^M)} - \sum_{m=0}^{r-1}\Big|P(e^{-i2m\pi/M}z)\Big|^2 \times \frac{1}{S(e^{-i2m\pi/M}z)}\Big).
\end{aligned}
\tag{4.15}
$$

From (4.14), (4.15) and the assumption $S(z) \geq 0$, it follows that

$$
\begin{aligned}
\det \mathcal{M}_r(z) \geq& \prod_{m=0}^{r-1} S(e^{-i2m\pi/M}z)S(z^M) \\
& \times \Big(\frac{1}{S(z^M)} - \sum_{m=0}^{M-1}\Big|P(e^{-i2m\pi/M}z)\Big|^2 \times \frac{1}{S(e^{-i2m\pi/M}z)}\Big) \\
=& \Big(\prod_{m=r}^{M-1} S(e^{-i2m\pi/M}z)\Big)^{-1} \det \mathcal{M}(z) \geq 0.
\end{aligned}
\tag{4.16}
$$

Thus, the upper-left square corner of $\mathcal{M}(z)$ of block size $r\times r$ has nonnegative determinants for any $1 \leq r \leq M$. Hence,

$$\mathcal{M}(z) \quad \text{is positive semi-definite on } |z| = 1. \tag{4.17}$$

Define $S_{m,m'}(z)$ and $\tilde{\mathcal{M}}(z)$ as in (4.4) and (4.6), respectively. Then it follows from $S(z) = S(z^{-1})$ that $S_{m,m'}(z^{-1}) = S_{m',m}(z)$, which implies that $\tilde{\mathcal{M}}(z^{-1}) = \tilde{\mathcal{M}}(z)^T$. Clearly, we have

$$\mathcal{P}_M(z^{-1})^T\mathcal{M}(z)\mathcal{P}_M(z) = \tilde{\mathcal{M}}(z^M), \tag{4.18}$$

where

$$\mathcal{P}_M(z) := \frac{1}{M}[e^{-i2mm'\pi/M}z^{m'}]_{0\leq m,m'\leq M-1} \tag{4.19}$$

is called a polyphase matrix. Hence, by (4.17) and (4.18), $\tilde{\mathcal{M}}(z)$ is positive semi-definite on $|z| = 1$. By Lemma 4.2, there exists an $M \times M$ matrix $\mathbf{R}(z)$ such that all entries of $\mathbf{R}(z)$ are Laurent polynomials with real coefficients, and

$$\mathbf{R}(z)^T\mathbf{R}(z^{-1}) = \tilde{\mathcal{M}}(z), \qquad |z| = 1. \tag{4.20}$$

If we write $\mathbf{R}(z) = [Q_{n,m}(z)]_{1\le n\le M, 0\le m\le M-1}$, then it follows from (4.20) and Lemma 4.3 that the Laurent polynomials

$$Q_n(z) := \sum_{m=0}^{M-1} z^m Q_{n,m}(z^M), \quad 1 \le n \le M, \tag{4.21}$$

satisfy the equation (3.2). ■

5. Tight Affine Frames with M Generators

For any given MRA generated by some compactly supported M-dilation scaling function ϕ, there are infinitely many Laurent functions S that satisfy the conditions (i)–(iii) in Theorem 4.1, and also infinitely many tight affine frames associate with ϕ. In this section, we choose the function S via eigenfunctions of the transfer operator to construct tight affine frames with M generators that have the maximum order of vanishing moments as governed by ϕ, as follows.

Theorem 5.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by a compactly supported M-dilation scaling function with order $L \ge 1$ as defined by its scaling symbol in* (1.7). *Then there exist M compactly supported functions $\Psi = \{\psi_1, \dots, \psi_M\} \subset V_1$ such that $\{\psi_1, \dots, \psi_M\}$ is a tight affine frame and that each of $\psi_n,\ n = 1, \dots, M$ has L vanishing moments.*

Proof. Let ϕ be an M-dilation scaling function with Laurent polynomial symbol $P(z)$ satisfying (1.7), where $P_0(z)$ is not divisible by $\frac{1}{M}(1 + z + \ldots + z^{M-1})$. By Proposition 2.1, there does not exist a nonzero polynomial $H(z)$ such that $H(z)$ is not a monomial and $H(z)P_0(z)$ is divisible by $H(z^M)$. We denote by ρ the spectral radius of the transfer operator $T_{|P_0|^2}$ on the space $\Pi_{\mathcal{N}}$, where $\mathcal{N}$ is chosen so large that $\Pi_{\mathcal{N}}$ is an invariant subspace of $T_{|P_0|^2}$. Then by Lemmas 2.4 and 2.6, we have

$$\rho < M^{2L} \quad \text{and} \quad T_{|P_0|^2} F = \rho F \tag{5.1}$$

for some positive Laurent polynomial $F(z) > 0$ on $|z| = 1$. Set

$$F_L(z) = \left|\frac{1-z}{2}\right|^{2L} F(z).$$

Then it follows from (2.11) and (5.1) that

$$T_{|P|^2} F_L = M^{-2L} \rho F_L \quad \text{and} \quad F_L(z) \ge 0 \quad \text{on } |z| = 1. \tag{5.2}$$

Next, consider the Laurent polynomial $\Phi(z)$ in (2.8). By Lemma 2.3, we have

$$T_{|P|^2} \Phi = \Phi. \tag{5.3}$$

We claim that there exists a Laurent polynomial $S(z)$ such that $S(1) = 1$,

$$S(z) = S(1/z), \tag{5.4}$$

and

$$\frac{1}{\Phi(z) + F_L(z)} \le S(z) \le \frac{1}{\Phi(z) + \gamma F_L(z)}, \qquad |z| = 1, \tag{5.5}$$

where $M^{-2L}\rho < \gamma < 1$. To construct $S(z)$, we first expand

$$\frac{1}{\Phi(z)} = \sum_{j=0}^{\infty} \sigma_j (2 - z - 1/z)^j$$

and let

$$S_0(z) := \sum_{j=0}^{L-1} \sigma_j (2 - z - 1/z)^j.$$

Then $S_0(z)$ is a symmetric Laurent polynomial that satisfies

$$S_0(1) = \frac{1}{\Phi(1)} = \frac{1}{\sum_{k\in\mathbf{Z}} |\hat{\phi}(2k\pi)|^2} = \frac{1}{|\hat{\phi}(0)|^2} = 1.$$

Let

$$r(z) := 2^{2L} \frac{S_0(z)\Phi(z) - 1}{(2 - z - 1/z)^L}.$$

It is easy to see that $r(z)$ is a symmetric Laurent polynomial. Since $F(z) > 0$ and $\Phi(z) > 0$ on $|z| = 1$, it follows by trigonometric approximation that there exists a symmetric Laurent polynomial $S_1(z)$, such that

$$\frac{\gamma F(z)}{\Phi(z)(\Phi(z) + \gamma F_L(z))} - \frac{r(z)}{\Phi(z)} \le S_1(z) \le \frac{F(z)}{\Phi(z)(\Phi(z) + F_L(z))} - \frac{r(z)}{\Phi(z)}, \quad |z| = 1.$$

Let

$$S(z) := S_0(z) - \left|\frac{1-z}{2}\right|^{2L} S_1(z).$$

Then $S(z)$ is a symmetric Laurent polynomial with $S(1) = 1$, and satisfies (5.5). Consequently $S(z) > 0$ on $|z| = 1$, and

$$\begin{aligned} \frac{1}{S(z)} - \Big(T_{|P|^2}\frac{1}{S}\Big)(z) &\ge \Phi(z) + \gamma F_L(z) - (T_{|P|^2}(\Phi + F_L))(z) \\ &= \Phi(z) + \gamma F_L(z) - \Phi(z) - M^{-2L}\rho F_L(z) \\ &= (\gamma - M^{-2L}\rho) F_L(z) \ge 0, \end{aligned} \tag{5.6}$$

where we have used (5.5) to obtain the first inequality, and (5.2) and (5.3) to achieve the first equality. It is easy to verify that

$$\begin{aligned}
&\prod_{m=0}^{M-1} S(e^{-i2m\pi/M}z) - S(z^M)\sum_{m=0}^{M-1}\Big(\Big|P(e^{-i2m\pi/M}z)\Big|^2 \prod_{i=0,i\neq m}^{M-1} S(e^{-i2m\pi/M}z)\Big)\\
&= \prod_{m=0}^{M-1} S(e^{-i2m\pi/M}z)\times S(z^M)\times\Big(\frac{1}{S(z^M)} - \Big(T_{|P|^2}\frac{1}{S}\Big)(z^M)\Big).
\end{aligned}$$

This together with (5.6) lead to the inequality (4.1). Finally, observe that (5.5) leads to

$$\frac{\gamma F_L(z)}{\Phi(z)(\Phi(z)+\gamma F_L(z))} \le \frac{1}{\Phi(z)} - S(z) \le \frac{F_L(z)}{\Phi(z)(\Phi(z)+F_L(z))}, \quad |z|=1,$$

which implies that

$$S(z) - \frac{1}{\Phi(z)} = O((1-z)^{2L}) \quad \text{as} \quad z \to 1,$$

and hence, together with (5.3), lead to

$$\begin{aligned}
S(z) - S(z^M)|P(z)|^2 &= \frac{1}{\Phi(z)} - \frac{|P(z)|^2}{\Phi(z^M)} + O((1-z)^{2L})\\
&= \frac{\Phi(z^M) - |P(z)|^2\Phi(z)}{\Phi(z)\Phi(z^M)} + O((1-z)^{2L})\\
&= \frac{\sum_{m=1}^{M-1}|P(e^{-i2m\pi/M}z)|^2\Phi(e^{-i2m\pi/M}z)}{\Phi(z)\Phi(z^M)} + O((1-z)^{2L})\\
&= O((1-z)^{2L}).
\end{aligned}$$

The last equality follows from the observation that $P(e^{-i2m\pi/M}z)$ is divisible to $(1-z)^L$ for $m = 1, \ldots, M-1$. Hence from (3.2), by putting $m = 0$, we have

$$|Q_n(z)| = O((1-z)^L) \quad \text{as} \quad z \to 1,$$

and that the tight frame generators Ψ have L vanishing moments, namely,

$$\int_{\mathbb{R}} x^l \psi_n(x)dx = 0, \qquad 0 \le l \le L-1 \quad \text{and } 1 \le n \le M.$$

This is the maximal order as governed by the scaling function ϕ. ■

6. Tight Affine Frames with $M-1$ Generators

We say that an $H(z)$ in the Wiener class $\mathcal{W}$ is an M-CQF (conjugate quadrature filter with dilation M) if $H(1)=1$ and

$$\sum_{m=0}^{M-1} \left|H(e^{-i2m\pi/M}z)\right|^2 = 1, \qquad |z|=1. \tag{6.1}$$

In this section, we characterize those M-dilation MRA's that allow compactly supported tight affine frames with $M-1$ generators.

Theorem 6.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by an M-dilation compactly supported scaling function ϕ with Laurent polynomial symbol $P(z)$. Then there exist compactly supported functions $\psi_1,\ldots,\psi_{M-1}\in V_1$ such that $\Psi=\{\psi_1,\ldots,\psi_{M-1}\}$ is a (normalized) tight affine frame, if and only if there exists a Laurent polynomial B such that $B(1)=1$, $B(z^M)/B(z)$ is a Laurent polynomial, and $B(z^M)P(z)/B(z)$ is an M-CQF. Furthermore, the functions $\psi_1,\ldots,\psi_{M-1}$ can be so chosen that their integer translates constitute a frame of the shift invariant space $W(\psi_1,\ldots,\psi_{M-1})$.*

For $M=2$, the above result on the existence of tight affine frames with one generator was established in [6], and the necessity is given in [21]. The frame property of $\psi_1,\ldots,\psi_{M-1}$ in the above theorem is new even for $M=2$.

Proof. First let us prove the sufficiency direction. Let a Laurent polynomial B be given as stated in the theorem. Clearly, the Laurent polynomial

$$G_0(z) = B(z^M)P(z)/B(z)$$

satisfies

$$\sum_{m=0}^{M-1} \left|G_0(e^{-i2m\pi/M}z)\right|^2 = 1.$$

By unitary matrix extension (cf. [20]), there exist Laurent polynomials $G_1,\ldots,G_{M-1}$ (with real coefficients if P and B have real coefficients), such that

$$\sum_{m=0}^{M-1} G_j(e^{-i2m\pi/M}z)G_{j'}(e^{i2m\pi/M}z^{-1}) = \delta_{jj'}, \quad 0\le j,j'\le M-1.$$

Hence, we also have

$$\sum_{m=0}^{M-1} G_m(e^{-i2j\pi/M}z)G_m(e^{i2j'\pi/M}z^{-1}) = \delta_{jj'}, \quad 0\le j,j'\le M-1. \tag{6.2}$$

By (6.2) and the assumption on $B(z)$, one can verify that the Laurent polynomial $S(z) := |B(z)|^2$ satisfies $S(1) = 1, S(z) \geq 0, S(1/z) = S(z)$ on $|z| = 1$, and

$$\begin{aligned}&\sum_{m=1}^{M-1} G_m(z)B(z)G_m(e^{i2s\pi/M}z^{-1})B(e^{i2s\pi/M}z^{-1})\\ =&\delta_{s0}S(z) - S(z^M)P(z)P(e^{i2s\pi/M}z^{-1}) \quad \forall\ 0 \leq s \leq M-1.\end{aligned}$$

Therefore, the functions $\psi_m, 1 \leq m \leq M-1$, defined by

$$\widehat{\psi}_m(\omega) = B(z)G_m(z)\hat{\phi}(\omega/M), \qquad z = e^{-i\omega/M}, \tag{6.3}$$

generate a tight affine frame by Theorem 3.1. Furthermore, their symbols $Q_m = BG_m$, $1 \leq m \leq M-1$, are Laurent polynomials with real coefficients.

Next we establish the necessity direction. Let $\psi_1, \ldots, \psi_{M-1}$ be a tight affine frame, defined by $\widehat{\psi}_m(\omega) = Q_m(z)\hat{\phi}(\omega/M)$, where $Q_1, \ldots, Q_{M-1}$ are Laurent polynomials. By Theorem 3.1, there exists a Laurent polynomial $S(z)$ that satisfies conditions (i) – (iii) in Theorem 3.1. Let $\mathcal{M}(z)$ be defined by (4.11). Then it follows from (4.10), with $N = M-1$, that the rank of $\mathcal{M}(z)$ is at most $M-1$, which leads to $\det \mathcal{M}(z) = 0$. Therefore, we have

$$\begin{aligned}&S(z^M)\sum_{m=0}^{M-1}\left(\left|P(e^{-i2m\pi/M}z)\right|^2 \prod_{0\leq m'\leq M-1, m'\neq m} S(e^{-i2m'\pi/M}z)\right)\\ =&\prod_{m=0}^{M-1} S(e^{-i2m\pi/M}z).\end{aligned} \tag{6.4}$$

A comparison of the degrees of the Laurent polynomials on both sides of equation (6.4) leads to the existence of a positive constant C such that

$$S(z^M) = C\prod_{m=0}^{M-1} S(e^{-i2m\pi/M}z) \tag{6.5}$$

and

$$\sum_{m=0}^{M-1}\left(\left|P(e^{-i2m\pi/M}z)\right|^2 \prod_{0\leq m'\leq M-1, m'\neq m} S(e^{-i2m'\pi/M}z)\right) = C^{-1}. \tag{6.6}$$

By (6.5) and (6.6), any Laurent polynomial $B(z)$ with $|B(z)|^2 = S(z)$ has the required properties, where the existence of the Laurent polynomial $B(z)$ follows from the (scalar-valued) Riesz Lemma. This completes the proof of the necessity direction.

Finally, we show that the collection of compactly supported functions $\psi_1, \ldots, \psi_{M-1}$ in (6.3) constitute a frame of the shift-invariant space $W(\psi_1, \ldots, \psi_{M-1})$. To this end, let

$$\mathcal{G}(z) = \left[G_m(e^{-i2s\pi/M}z)\right]_{0\leq m\leq M-1, 0\leq s\leq M-1}$$

be the unitary matrix found in (6.2). Obviously, the rank of $\mathcal{G}(z)$ is M on $|z| = 1$. By Proposition 2.7, it suffices to verify that

$$\operatorname{rank} A(z) = M - 1, \qquad \text{on } |z| = 1, \tag{6.7}$$

where

$$A(z) = \Big[B(e^{-i2s\pi/M}z)G_m(e^{-i2s\pi/M}z)\Big]_{1\le m\le M-1, 0\le s\le M-1}.$$

Note that we have $A = \mathcal{G}_1\mathcal{B}$, where $\mathcal{G}_1$ is obtained from $\mathcal{G}$ by leaving out its first row, and $\mathcal{B} := \operatorname{diag}(B(z), B(e^{-i2\pi/M}z), \ldots, B(e^{-i2(M-1)\pi/M}z))$.

If z_0 is so chosen that all the diagonal entries of $\mathcal{B}(z_0)$ are nonzero, then

$$M - 1 \ge \operatorname{rank} A(z) = \operatorname{rank} \mathcal{G}_1(z) = M - 1. \tag{6.8}$$

For any z, $|z| = 1$, with $B(e^{-i2s_0\pi/M}z) = 0$ for some $0 \le s_0 \le M-1$, we claim that

$$B(e^{-i2s\pi/M}z) \ne 0, \qquad \text{if } s - s_0 \notin M\mathbf{Z}. \tag{6.9}$$

Suppose, on the contrary, that $B(e^{-i2s_1\pi/M}z) = 0$ for some $s_1 \in \mathbf{Z}$ with $s_1 - s_0 \notin M\mathbf{Z}$. Then there exists a z_0 with $|z_0| = 1$ such that $B(z_0^M)/B(e^{-i2s\pi/M}z_0) = 0$ for all $s \in \mathbf{Z}$ in view of (6.5) and $|B(z)|^2 = S(z)$, which violates the M-CQF condition for the filter $B(z^M)P(z)/B(z)$. Therefore, the first row of the unitary matrix $\mathcal{G}(z)$ is given by

$$(0, \ldots, 0, G_0(e^{-i2s_0\pi/M}z), 0, \ldots, 0),$$

where $G_0(e^{-i2s_0\pi/M}z) \ne 0$. Hence, the adjoint factor $\det \mathcal{G}_{1,s_0}$ must be nonzero, where $\mathcal{G}_{1,s_0}$ denotes the matrix obtained by leaving out the first row and the s_0-th column of $\mathcal{G}$. The matrix

$$A'(z) := \Big[B(e^{-i2s\pi/M}z)G_m(e^{-i2s\pi/M}z)\Big]_{1\le m\le M-1, 0\le s\ne s_0\le M-1}$$

is a sub matrix of $A(z)$. Moreover, we have $A'(z) = \mathcal{G}_{1,s_0}\mathcal{B}_1$, where $\mathcal{B}_1$ is the diagonal matrix with entries $(B(z), \ldots, B(e^{-i2(s_0-1)\pi/M}z), B(e^{-i2(s_0+1)\pi/M}z), \ldots, B(e^{-i2(M-1)\pi/M}z))$ in its diagonal. By (6.9), the matrix $\mathcal{B}_1$ is invertible, and we obtain

$$M - 1 \ge \operatorname{rank} A(z) \ge \operatorname{rank} A'(z) = \operatorname{rank} \mathcal{G}_{1,s_0}(z) = M - 1. \tag{6.10}$$

Therefore (6.7) follows from (6.8) and (6.10). ■

7. Examples

In this section, tight affine frames associated with the linear Cardinal spline multiresolution approximation are constructed with dilation factors $M = 3$ and 4. The scaling function ϕ is the normalized linear B-spline defined by $\phi(x) = 1 - |x-1|$ for $|x-1| \le 1$ and $\phi(x) = 0$ otherwise.

Example 1. We consider dilation factor $M = 3$ in this example and set $z = e^{-i\omega/3}$. The symbol of the scaling function is therefore $P(z) = \frac{1}{9}(1 + z + z^2)^2$. The function $S(z) = 1 + \frac{1}{6}(2 - z - z^{-1})$ is chosen in order to yield 2 vanishing moments for each ψ_n, $1 \le n \le 3$. (The function S could be the same function as in the case $M = 2$ in [6, 12] and $M = 4$ in Example 2 below.) In the case under consideration, $S(z) - S(z^3)P(z)P(1/z)$ is a Laurent polynomial multiple of $(1-z)^4$. Our goal is to construct Laurent polynomials

$$Q_n(z) = (1-z)^2 q_n(z), \quad n = 1, 2, 3, \tag{7.1}$$

where q_n are again Laurent polynomials. If we set $\zeta := e^{i2\pi/3} = -1/2 + \sqrt{3}/2i$, then (3.2) can be put in matrix form as

$$\begin{aligned}&\begin{bmatrix} Q_1(z) & Q_1(\zeta z) & Q_1(\zeta^2 z) \\ Q_2(z) & Q_2(\zeta z) & Q_2(\zeta^2 z) \\ Q_3(z) & Q_3(\zeta z) & Q_3(\zeta^2 z) \end{bmatrix}^* \begin{bmatrix} Q_1(z) & Q_1(\zeta z) & Q_1(\zeta^2 z) \\ Q_2(z) & Q_2(\zeta z) & Q_2(\zeta^2 z) \\ Q_3(z) & Q_3(\zeta z) & Q_3(\zeta^2 z) \end{bmatrix} \\ &= \begin{bmatrix} S(z) & & \\ & S(\zeta z) & \\ & & S(\zeta^2 z)) \end{bmatrix} - S(z^3) \begin{bmatrix} \overline{P(z)} \\ \overline{P(\zeta z)} \\ \overline{P(\zeta^2 z)} \end{bmatrix} [\, P(z) \quad P(\zeta z) \quad P(\zeta^2 z) \,] .\end{aligned} \tag{7.2}$$

The notation $*$ denotes, as usual, complex conjugate transposition. Since $|z| = 1$, we have $\overline{z} = 1/z$, $\overline{\zeta z} = \zeta^2/z$, and $\overline{\zeta^2 z} = \zeta/z$. We can multiply both sides of (7.2) by certain appropriate diagonal matrices in order to eliminate the factors $(1-z)^2$ in (7.1). This leads to

$$\begin{aligned}&\begin{bmatrix} q_1(z) & q_1(\zeta z) & q_1(\zeta^2 z) \\ q_2(z) & q_2(\zeta z) & q_2(\zeta^2 z) \\ q_3(z) & q_3(\zeta z) & q_3(\zeta^2 z) \end{bmatrix}^* \begin{bmatrix} q_1(z) & q_1(\zeta z) & q_1(\zeta^2 z) \\ q_2(z) & q_2(\zeta z) & q_2(\zeta^2 z) \\ q_3(z) & q_3(\zeta z) & q_3(\zeta^2 z) \end{bmatrix} \\ &= \begin{bmatrix} \frac{S(z) - S(z^3)|P(z)|^2}{|1-z|^4} & -\frac{S(z^3)\overline{P(z)}P(\zeta z)}{(1-1/z)^2(1-\zeta z)^2} & -\frac{S(z^3)\overline{P(z)}P(\zeta^2 z)}{(1-1/z)^2(1-\zeta^2 z)^2} \\ -\frac{S(z^3)\overline{P(\zeta z)}P(z)}{(1-\zeta^2/z)^2(1-z)^2} & \frac{S(\zeta z) - S(z^3)|P(\zeta z)|^2}{|1-\zeta z|^4} & -\frac{S(z^3)\overline{P(\zeta z)}P(\zeta^2 z)}{(1-\zeta^2/z)^2(1-\zeta^2 z)^2} \\ -\frac{S(z^3)\overline{P(\zeta^2 z)}P(z)}{(1-\zeta/z)^2(1-z)^2} & -\frac{S(z^3)\overline{P(\zeta^2 z)}P(\zeta z)}{(1-\zeta/z)^2(1-\zeta z)^2} & \frac{S(\zeta^2 z) - S(z^3)|P(\zeta^2 z)|^2}{|1-\zeta^2 z|^4} \end{bmatrix} .\end{aligned} \tag{7.3}$$

Note that since $P(z) = \frac{1}{9}(1 - \zeta z)^2(1 - \zeta^2 z)^2$, the matrix on the right-hand side of (7.3) is a Laurent polynomial matrix. Let $q_{nj}(z)$, $j = 0, 1, 2$, denote the polyphase components of

$q_n(z)$, $n = 1,2,3$; that is, $q_n(z) = q_{n0}(z^3) + zq_{n1}(z^3) + z^2 q_{n2}(z^3)$. The polyphase matrix

$$\mathcal{P}_3(z) = \frac{1}{3}\begin{bmatrix} 1 & z^{-1} & z^{-2} \\ 1 & \zeta^2 z^{-1} & \zeta z^{-2} \\ 1 & \zeta z^{-1} & \zeta^2 z^{-2} \end{bmatrix},$$

as defined in (4.19), reveals the polyphase components by considering the product

$$(q_n(z), q_n(\zeta z), q_n(\zeta^2 z))\mathcal{P}_3(z) = (q_{n0}(z^3), q_{n1}(z^3), q_{n2}(z^3)), \qquad n = 1,2,3.$$

The polyphase decomposition of (7.3) is then obtained by multiplying $\mathcal{P}_3(z)$ from the right, $\mathcal{P}_3(z)^*$ from the left, and replacing z^3 by z. This gives the explicit form

$$\begin{bmatrix} q_{10}(z) & q_{11}(z) & q_{12}(z) \\ q_{20}(z) & q_{21}(z) & q_{22}(z) \\ q_{30}(z) & q_{31}(z) & q_{32}(z) \end{bmatrix}^* \begin{bmatrix} q_{10}(z) & q_{11}(z) & q_{12}(z) \\ q_{20}(z) & q_{21}(z) & q_{22}(z) \\ q_{30}(z) & q_{31}(z) & q_{32}(z) \end{bmatrix} \tag{7.4}$$

$$= \frac{1}{243}\begin{bmatrix} 5z + 68 + 5z^{-1} & 2z + 67/2 + 20z^{-1} & 1/2z + 14 + 50z^{-1} \\ 20z + 67/2 + 2z^{-1} & 8z + 44 + 8z^{-1} & 2z + 67/2 + 20z^{-1} \\ 50z + 14 + 1/2z^{-1} & 20z + 67/2 + 2z^{-1} & 5z + 68 + 5z^{-1} \end{bmatrix},$$

from which the Laurent polynomials q_{nj} can be constructed as follows. We denote the matrix on the right-hand side of (7.3) by $\mathcal{M}(z)$. Let $E := E_1E_2E_3$, where E_1, E_2 and E_3 are the elementary matrices

$$E_1 := \begin{bmatrix} 1 & 0 & 0 \\ -11/4 & 1 & 0 \\ 1 & -2/5 & 1 \end{bmatrix}, E_2 := \begin{bmatrix} 1 & 0 & -50/1821 \\ 0 & 1 & -995/3642 \\ 0 & 0 & 1 \end{bmatrix}, E_3 := \begin{bmatrix} 1 & 37/95z^{-1} & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$E^*\mathcal{M}(z)E = \frac{1}{243}\begin{bmatrix} 513/4 & 0 & 0 \\ 0 & 16389/1900 & 18 \\ 0 & 18 & 50/607z + 40400/607 + 50/607z^{-1} \end{bmatrix}.$$

The last matrix can be easily written as a product $R(z^{-1})^T R(z)$, where

$$R(z) = \begin{bmatrix} 3\sqrt{57}/2 & 0 & 0 \\ 0 & 3\sqrt{34599}/190 & 20\sqrt{34599}/607 \\ 0 & 0 & (5\sqrt{107439}/607 + 25\sqrt{4249}/607)(z + 176 - 5\sqrt{1239}) \end{bmatrix}.$$

It follows that $243\mathcal{M}(z)$ can be written as $\mathcal{R}(z^{-1})^T\mathcal{R}(z)$, where $\mathcal{R} = RE^{-1}$ has the form

$$\mathcal{R}(z) := \begin{bmatrix} -9\sqrt{57}/76 & -12\sqrt{57}/19 & -9\sqrt{57}/76 \\ \alpha(2023z + 152)/46132 & 2\alpha(101z + 76)/11533 & \alpha(199z + 1520)/46132 \\ \beta(176 - \gamma + z)/1214 & 2\beta(176 - \gamma + z)/607 & 5\beta(176 - \gamma + z)/607 \end{bmatrix}$$

and $\alpha = \sqrt{34599}$, $\beta = \sqrt{607}(\sqrt{177} + 5\sqrt{7})$, $\gamma = 5\sqrt{1239}$. Notice that $243\mathcal{M}(z)$ can also be written as $\mathcal{R}^T(z^{-1})\mathcal{O}^T\mathcal{O}\mathcal{R}(z)$ for any orthogonal constant matrix $\mathcal{O}$. We may choose an appropriate $\mathcal{O}$ to reduce the support length of some ψ_n. Here, we consider

$$\mathcal{O} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \qquad c = 380\sqrt{607}(\sqrt{177} + 5\sqrt{7})/r,\ s = 199\sqrt{34599}/r,$$

and $r := \sqrt{32223236599 + 876508000\sqrt{1239}}$. These are exact values that are computed with **Maple**. Finally we obtain

$$\begin{bmatrix} q_{10}(z) & q_{11}(z) & q_{12}(z) \\ q_{20}(z) & q_{21}(z) & q_{22}(z) \\ q_{30}(z) & q_{31}(z) & q_{32}(z) \end{bmatrix} = \frac{1}{9\sqrt{3}}\mathcal{O}\mathcal{R}(z)$$

for the polyphase components of q_n, $1 \le n \le 3$, in (7.2). The symbols Q_n, $1 \le n \le 3$, are

$$Q_1(z) = -(1-z)^2(.0574 + .3059z + .0574z^2),$$
$$Q_2(z) = (1-z)^2(.0389 + .1555z + .3887z^2 + .5125z^3 + .1863z^4),$$
$$Q_3(z) = (1-z)^2(.0059 + .0236z + .0589z^2 + .1113z^3 + .1675z^4 + .3492z^5).$$

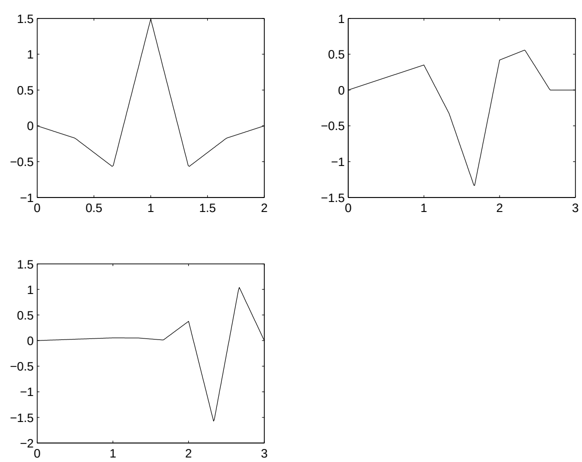

Fig. 1. A linear B-spline tight frame with dilation factor $M = 3$ and 3 generators, each having 2 vanishing moments

Example 2. For dilation by $M = 4$, the symbol of the scaling function is $P(z) = \frac{1}{16}(1 + z + z^2 + z^3)^2$. The VMR function $S(z)$we will use is still $1 + \frac{1}{6}(2 - z - z^{-1})$ and it assures

2 vanishing moments. By following the same approach as above, we can find that

$$
\begin{aligned}
Q_1(z) &= -(1-z)^2(.0265 + .1061z + .3339z^2 + .0907z^3 + .0073z^4) \\
Q_2(z) &= (1-z)^2(.0145 + .0581z + .1508z^2 + .3038z^3 - .0179z^4) \\
Q_3(z) &= (1-z)^2(.0188 + .0751z + .1933z^2 + .3556z^3 + .3671z^4) \\
Q_4(z) &= -(1-z)^2(.0209z^{-4} + .0835z^{-3} + .2087z^{-2} + .4173z^{-1} + .5942 \\
&\qquad + .6240z + .3120z^2 + .1248z^3 + .0312z^4)
\end{aligned}
$$

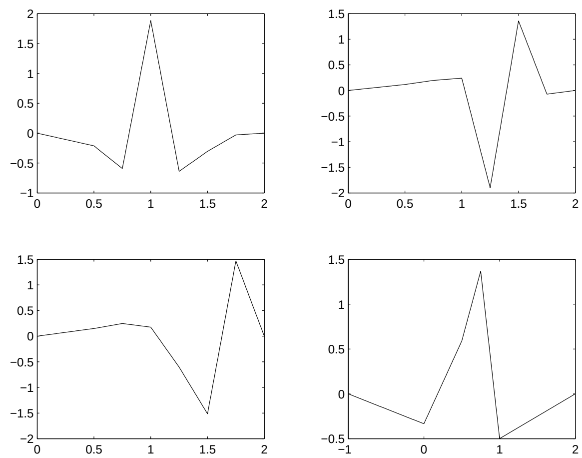

Fig. 2. A linear B-spline tight frame with dilation factor $M = 4$ and 4 generators, each having 2 vanishing moments

Final Remark. After this manuscript was completed, we were aware of the work [15] that also considers a generalization of [6, 11, 12]. In [15], the generalization is to bi-frames instead of tight frames as in our paper. A significant difficulty in the treatment of tight frames is that a matrix-valued Riesz Lemma is required to find the "square root" of a positive semi-definite matrix of Laurent polynomials that remains to be a matrix of Laurent polynomials. Also restrictions on the vanishing moment recovery functions for tight frames and bi-frames are different. On the other hand, the generalization in [15] is to multiwavelet (i.e. vector-valued) bi-frames with arbitrary integer dilation.

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