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Generalized Strang-Fix condition for scattered data quasi-interpolation

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Quasi- interpolation is very useful in the study of the approximation theory and its applications, since the method can yield solutions directly and does not require solving any linear system of equations. However, quasi- interpolation is usually discussed only for gridded data in the literature. In this paper we shall introduce a generalized Strang- Fix condition, which is related to non- stationary quasi- interpolation. Based on the discussion of the generalized Strang- Fix condition we shall generalize our quasi- interpolation scheme for multivariate scattered data, too.

Key Words: Quasi- Interpolation, Strang- Fix Condition, Scattered Data Approximation, Shift- Invariant Space, Radial Basis Interpolation.
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1. INTRODUCTION

Quasi- interpolation in its standard form takes values $f(jh),\ j \in \mathbb{Z}^d$ of a d- variate function f on a grid with spacing h and a set of given basis functions $\Phi_{j,h}(x)$ to construct an approximant of f via linear combination

$$\sum f(jh)\Phi_{j,h}(x) \sim f(x). \tag{1.1}$$

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The advantage of quasi- interpolation is that one can evaluate the approximant directly, and does not require solving any linear systems of equations. The earliest case of quasi- interpolation is perhaps Bernstein's approximation, which uses the Bernstein polynomials

$$B_j^n(x) = \binom{n}{j} x^j (1-x)^{n-j}$$

to build a quasi- interpolant of an univariate function f on $[0,1]$ via

$$\sum_{j=0}^{n} f(\frac{j}{n}) B_j^n(x), \qquad x \in [0,1].$$

This is a basic scheme in Approximation Theory and Functional Analysis, but in addition it is widely used in Computer Aided Geometric Design under the names of Bezièr and de Casteljau. Another well- known quasi-interpolation scheme is formed by the reconstruction of bandlimited functions via the Whittaker- Shannon sampling series. Finally, there is the well- known B- spline series, which is included in any computer software for representation of curves and surfaces. We refer to [1, 2, 3, 4, 7, 8, 9, 10, 12, 13, 15, 17, 27] and the references therein for more details of quasi-interpolation and related topics.

We will begin our study with Schoenberg's model [17]

$$\sum f(jh)\Phi(\frac{x}{h} - j) \sim f(x), \qquad x \in I\!R^d \tag{1.2}$$

which is a simplified form of (1.1) where the functions $\Phi_{j,h}(x)$ are scaled shifts of a single kernel function Φ on $I\!R^d$. Usually, this kernel is assumed to be continuous, even, and Fourier- transformable with a real- valued Fourier transform. This model is used, for instance, with the Shannon sampling theorem and the B- spline series.

Thanks to Strang and Fix [22], the convergence order result

$$\| \sum f(jh)\Phi(\frac{x}{h} - j) - f(x)\|_\infty \le \mathcal{O}(h)^l \tag{1.3}$$

for $h \to 0$ holds for any sufficiently smooth function f, if and only if

$$|\hat{\Phi}(w) - 1| \le \mathcal{O}(w)^l, \quad w \to 0, \tag{1.4}$$

and

$$|\hat{\Phi}(2\pi j + w)| \le \mathcal{O}(w)^l \quad w \to 0 \tag{1.5}$$

hold. The paper [19] showed that the condition (1.4) can easily be satisfied by modifying a given Φ. In particular, by a finite linear combination of shifts of the function Φ one can satisfy the condition (1.4) if $\hat{\Phi} \in C^l$ and $\hat{\Phi}(0) \neq 0$.

A lot of functions satisfy the Strang- Fix conditions, such as the uniform symmetric B- Spline and tensor product B- Splines. For conditionally positive definite radial functions such as multiquadrics and thin- plate splines, one can satisfy the Strang- Fix conditions too by taking finite linear combinations of the shifts of the function [5][6].

However, all conditionally positive definite radial functions are unbounded and not compactly supported. A negative result showed in [26] proves that no finite linear combinations of shifts of compactly supported strictly positive definite radial functions satisfy the Strang- Fix conditions. A similar negative result concerning positive definite radial functions like the Gaussian is in M.D. Buhmann's dissertation [5]. Among other things, this paper overcomes these problems by a generalization of both the quasi- interpolation scheme and the Strang- Fix conditions.

The advantages of radial basis functions [16, 18, 23, 25] are revealed especially in the case of multivariate approximation, where the computation of the basis is cheap. There one uses a single univariate function $\phi : I\!R_+ \to I\!R$ and scattered points $x_j \in I\!R^d$, called centers, to build shifted basis functions $\Phi_j(x) = \phi(\|x - x_j\|)$. For interpolation, one takes the centers exactly at the data points, and then the interpolation problem is uniquely solvable, provided that the function $\Phi(\cdot) = \phi(\|\cdot\|)$ is positive definite. Interpolation by radial basis functions has a well- developed theory, and it is a very powerful tool to approximate smooth functions. But since it needs to solve large systems of linear equations with possibly bad condition, it is a very interesting problem to go over to quasi- interpolation, especially in the case of compactly supported radial kernels.

A more general approach to quasi- interpolation proceeds via the scheme

$$\sum \lambda_j(f)\Phi(x - jh)$$

where $\lambda_j(f)$ are linear functionals of f. The order of approximation of such a quasi- interpolation scheme can be obtained via the theory of shift-invariant spaces, which is covered by various articles of de Boor, DeVore and Ron (e.g. [3, 4]). However, this scheme usually requires some additional work to get the values $\lambda_j(f)$, which need not consist of finitely many function values only, but can possibly require some integration. However, the scheme would be powerful, if the $\lambda_j(f)$ were readily available. Another problem is how to generalize this quasi- interpolation scheme to scattered data. Buhmann etc [6] discussed quasi- interpolation with radial basis on scattered centers. Dyn and Ron [8] used a two- step algorithm to construct a quasi- interpolant at first for gridded data followed by an approximation of the interpolant by radial basis functions based on scattered centers. Yoon [27] has put this work on a more general basis. Some related topics are discussed in [7] by Dyn etc too. However, the coefficients of the schemes are linear functionals of the function f too, but not in terms of the given pointwise function value data as in Schoenberg's scheme. Altogether, these schemes use radial basis functions with scattered centers, but not for scattered data. We do not want to study this topic here, because we shall stick to Schoenberg's model (1.2) due to its simplicity. In the end, we shall generalize Schoenberg's quasi- interpolation to scattered data, but still require point- evaluation data only.

By using the results of non- stationary quasi- interpolation shown in [13], we could construct quasi- interpolants with conditionally positive definite radial functions as kernels. This is due to the fact that one can construct linear combinations of shifts of conditionally positive definite functions to satisfy the Strang- Fix conditions. However, conditionally positive definite radial functions are not compactly supported. Furthermore, this discussion would be still in the context of classical Strang- Fix conditions [2, 11].

In this paper we will show a generalized Strang- Fix condition using a scaled variation Φ_h of a fixed given kernel Φ. Under the hypotheses $\hat{\Phi}(w) \in C^l$ and $\hat{\Phi}(0) \neq 0$ we can get a finite linear combination $\Psi(x) = \sum a_j \Phi(x-j)$ of shifts of the function Φ such that the scales Ψ_h of the function Ψ serve as a kernel for non- stationary quasi- interpolation with good convergence properties without satisfying the classical Strang- Fix conditions.

Li and Micchelli [15] have also proposed a non- stationary quasi- interpolant where the kernel is constructed by scales of some given function. In relation to our interpolant (3.7) they discuss the case $p = 1/k,\ h = 1/n$. We shall generalize their quasi- interpolation scheme and find out the optimal p in (3.7). This will yield an optimal order of convergence for such a quasi- interpolant. Finally, we construct quasi- interpolants for multivariate scattered data (but not scattered centers) using radial basis function. However, these results are far from complete and provide just a starting point for further investigation.

2. PARTITIONS OF UNITY

Before we go into details of quasi- interpolation, we will first introduce the concept of partition of unity.

Definition 1. A compactly supported function $\Phi : I\!R^d \to I\!R$ provides a *partition of unity* on the integer lattice, if

$$\sum_{j \in \mathbb{Z}^d} \Phi(x-j) = 1$$

holds for all $x \in I\!R^d$.

This is sufficient to yield a simple convergence result:

THEOREM 1. *For differentiable functions* $f : I\!R^d \to I\!R$ *with a uniform bound* $\|f'\|_{\infty,I\!R^d} < \infty$ *we have*

$$\|\sum f(jh)\Phi(\frac{\cdot}{h} - j) \to f(\cdot)\|_{\infty,I\!R^d} \leq h \cdot C(d,\Phi) \cdot \|f'\|_{\infty,I\!R^d},$$

if Φ *is bounded and provides a partition of unity.*

Proof: Assume that the radius of support of the function Φ is $R > 0$. Then

$$\begin{aligned} & |\sum f(jh)\Phi(\frac{x}{h} - j) - f(x)| \\ = & |\sum [f(jh) - f(x)]\Phi(\frac{x}{h} - j)| \\ = & |\sum_{|x-jh|\leq Rh} [f(jh) - f(x)]\Phi(\frac{x}{h} - j)| \\ \leq & c(d)\|f'\|_\infty \|\Phi\|_\infty Rh \end{aligned} \tag{2.6}$$

for small h, because the number of lattice points in the ball is proportional to $I\!R^d$ with just a dimension- dependent constant. ∎

Furthermore, if Φ is nonnegative, we can even get

$$|\sum f(jh)\Phi(\frac{x}{h} - j) - f(x)| \leq \|f'\|_\infty Rh.$$

The condition that the function Φ should be compactly supported is not always necessary. In fact, a fast decay of the function Φ is required only.

The above condition for the convergence of a quasi- interpolation scheme based on a partition of unity is stronger than the classical Strang- Fix conditions, since the latter are necessary and sufficient. However, the concept of a partition of unity can be easily generalized to scattered data. In fact, for any compactly supported function $\Phi \geq 0$ the function

$$\Psi_j(x) = \frac{\Phi(\frac{x - x_j}{h})}{\sum_k \Phi(\frac{x - x_k}{h})}$$

is nonnegative and satisfies a generalized partition of unity condition for scattered data points x_j, provided that the point density

$$h = \max_x \min_j \|x - x_j\|$$

is small enough. By suitable storage and retrieval techniques for scattered data one can evaluate such functions efficiently. The proof of convergence of

$$\sum f(x_j)\Psi_j(x) \sim f(x) \quad \text{as} \quad h \to 0$$

then proceeds along the same lines as (2.6). The first ingredient here is that partitions of unity always reproduce constants, i.e. they enjoy a polynomial reproduction property. Furthermore, by either nonnegativity or compact support they are locally stable. These two properties together yield convergence. Additional material on local methods with stable polynomial reproduction can be found in [20], section 9.

3. GENERALIZED STRANG- FIX CONDITION

Our final quasi- interpolation operator for gridded data will be

$$I(f)(x) \quad := \quad (h^p)^d \sum f(jh)\Phi\left(h^p\left(\frac{x}{h}-j\right)\right). \tag{3.7}$$

It is composed of linear combinations of the shifts and scales of the function Φ, carrying the advantages of Schoenberg's scheme over to the new quasi-interpolation. However, we want to explain its background first and relate it to the Strang- Fix conditions.

Employing the idea of the partition of unity and the idea of a non- stationary quasi- interpolation from a shift- invariant space, we can expect that

$$\sum f(jh)\Phi_h(\frac{x}{h}-j) \to f(x)$$

uniformly as $h \to 0$, if

$$\sum_j \Phi_h(x-j) \neq 1 \quad \text{but} \quad \to 1 \tag{3.8}$$

uniformly. Furthermore, we take Φ_h to be scaled versions

$$\Phi_h(x) := (h^p)^d \Phi(h^p x) \tag{3.9}$$

of a single given function Φ for computational reasons. To satisfy the condition (3.8) we assume

$$\int_{\mathbb{R}^d} \Phi(x)dx = 1 \tag{3.10}$$

and simply approximate this by a Riemannian sum over gridded values with step h^p. Then we should get (3.8) in the form

$$(h^p)^d \sum \Phi(x-jh^p) \to \int_{\mathbb{R}^d} \Phi(x)dx = 1,$$

for any x.

These observations enter into the quasi- interpolation operator (3.7), and the numerical integration argument is made precise by Theorem 4 in Appendix A.

We see that the condition (3.10) is equivalent to $\hat{\Phi}(0) = 1$, and is just the first part (1.4) of the Strang- Fix condition.

DEFINITION 2. We say that a function Φ satisfies the *generalized Strang-Fix condition*, if it satisfies (3.10), i.e. only the first part (1.4) of the classical Strang- Fix condition.

From the discussion up to this point it is clear that there is no reason why one should require the additional part of the classical Strang- Fix conditions.

4. ERROR ESTIMATES

The new quasi- interpolant (3.7) can be viewed as a numerical integration based on a rectangular rule with stepsize h^p of the integral

$$\int f(h^q t)\Phi(\frac{x}{h^q} - t)dt, \; q := 1 - p.$$

The error estimates of the new quasi- interpolant are partially based on the error estimates for the numerical integration. For the latter we refer to the appendix A at the end of the paper. There we also see that taking other rules like Simpson's will not improve the results. This can also be understood from integration of periodic functions and the analysis of Romberg integration via Richardson extrapolation of integration by trapezoidal sums. There, the lower order error terms come from the boundary only, if the integrand is smooth enough.

Now we go back to the quasi- interpolant (3.7) and split the error analysis in two parts. The integration error is

$$\|(h^p)^d \sum f(jh)\Phi(\frac{x}{h^q} - jh^p) - \int f(h^q t)\Phi(\frac{x}{h^q} - t)dt\|_\infty, \tag{4.11}$$

and we write the integral as a convolution at x/h^q of the functions $f(h^q \cdot)$ and Φ, i.e.

$$\int f(h^q t)\Phi(\frac{x}{h^q} - t)dt = \left(\widehat{f(h^q \cdot)}\hat{\Phi}\right)^\vee \left(\frac{x}{h^q}\right)$$

to define

$$\| \left(\widehat{f(h^q \cdot)}\hat{\Phi}\right)^\vee \left(\frac{x}{h^q}\right) - f(x)\|_\infty \tag{4.12}$$

as the second part of the error.

We start with the first part of the error and assume $|\Phi(x)| < o(1 + |x|)^{-s-d}$ and the Fourier transform $\hat{\Phi} \in C^s$ in what follows (these two condition is almost equivalent, here $|\Phi(x)| < o(1 + |x|)^{-s-d}$ means there is a small $\epsilon > 0$ that $|\Phi(x)| < \mathcal{O}(1+|x|)^{-s-d-\epsilon})$. By using the result in [19] we can get a finite linear combination $\Psi = \sum a_j \Phi(x-j)$ such that

$$|\hat{\Psi}(w) - 1| \leq \mathcal{O}(w)^s \quad \text{as} \quad w \to 0. \tag{4.13}$$

For simplicity, we now assume that the function Φ itself satisfies (4.13). Furthermore, if $f \in C^v$ and $\Phi \in C^u$ with $u \geq v$ and the derivatives of order v of the function f as well as the derivatives of order u of the function Φ are all globally bounded and absolutely integrable, then the discussion in appendix A concerning integration with stepsize $= h^p$ yields a bound of the form $\mathcal{O}(h^{\frac{svp}{s+d}})$ for $h \to 0$ for the first error. Note that x/h^q can be taken as a parameter, and the error estimates of Theorem 4 and Theorem 5 in Appendix A are uniform.

Remark 1. The condition that the function f should be bounded is not always necessary. The quasi- interpolant (3.7) is even valid for functions f with polynomial growth of order $s-1$. However, the order of approximation will be decreased in this case, since the estimation of the first part of the error depends on the order of the decay of the integrand.

To estimate the second part of the error we have

$$\begin{aligned}
& \left\| \left(\widehat{f(h^q\cdot)}\hat{\Phi} \right)^{\vee} \left(\tfrac{x}{h^q}\right) - f(x) \right\|_\infty \\
= & \left\| \int e^{-ixw} \hat{f}(w)(\hat{\Phi}(h^q w) - 1) dw \right\|_\infty \\
\leq & \left\| \int_{|w|<h^{-r}} e^{-ixw} \hat{f}(w)(\hat{\Phi}(h^q w) - 1) dw \right\|_\infty \\
& + \left\| \int_{|w|>h^{-r}} e^{-ixw} \hat{f}(w)(\hat{\Phi}(h^q w) - 1) dw \right\|_\infty \\
\leq & \|\hat{f}\|_1 h^{s(q-r)} + (\|\hat{\Phi}\|_\infty + 1) h^{rv} \\
\leq & \mathcal{O}(h^{\frac{sqv}{v+s}})
\end{aligned}$$

up to constant factors, if we set $r = \frac{qs}{v+s}$ to satisfy $rv = s(q-r)$.

Summarizing the analysis above we get

$$\begin{aligned}
& |h^p \sum f(jh) \Phi(\tfrac{x}{h^q} - jh^p) - f(x)| \\
\leq & \mathcal{O}(h^{\frac{svp}{s+d}}) + \mathcal{O}(h^{\frac{svq}{v+s}}) \\
\leq & \mathcal{O}(h^{\frac{sv}{(2s+v+d)}}),
\end{aligned}$$

if we choose $p = \frac{(s+d)}{(2s+v+d)}$ to satisfy $\frac{svp}{s+d} = \frac{svq}{v+s}$. Thus we have got an optimal choice of the parameter p in the quasi- interpolant (3.7).

THEOREM 2. *If $f \in C^v(I\!R^d)$ and $|\hat{f}(t)| < o(1+|t|)^{-v-d}$, both $\|f\|_\infty$ and $\|\hat{f}\|_{L_1}$ are bounded, if the kernel $\Phi \in C^u(I\!R^d)$ $(u \geq v)$ has decay $|\Phi| < o(1+\|x\|)^{-s-d}$, $\hat{\Phi} \in C^s$ and $\int \Phi(x)dx \neq 0$, then we can put $p = \frac{(s+d)}{(2s+v+d)}$ to construct a non- - stationary quasi- interpolant with the error estimate*

$$|C_\Psi(h^p)^d \sum f(jh)\Psi(\frac{x}{h^q} - jh^p) - f(x)| \leq \mathcal{O}(h^{\frac{sv}{2s+v+d}}),$$

where $C_\Psi^{-1} = \int \Psi(x)dx \neq 0$, and where the function Ψ is a finite linear combination of scaled shifts of the function Φ. ∎

Remark 2. Comparing with the classical Strang- Fix conditions, $\hat{\Phi}(0) \neq 0$ is now the only condition for the convergence of the new quasi- interpolant. The total approximation order depends in principle only on the continuity of the functions f, Φ and $\hat{\Phi}$. In this sense the new condition is therefore weaker than the classical Strang- Fix conditions.

Remark 3. The classical Strang- Fix conditions are necessary and sufficient. De Boor and Jia even showed that $\sigma_h = \text{span}\{\Phi(\frac{x}{h} - j)\}$ has local approximation order k if and only if some $\Psi \in \sigma_1$ satisfies the Strang- Fix conditions of order k (see [2] Theorem 5.4). What we have done here is different, because instead of the space σ_h we use $\text{span}\{\Phi(\frac{x}{h^q} - jh^p)\}$. In fact, $h^{pd}\Phi(h^p x)$ does not always satisfy the classical Strang- Fix conditions but satisfies them asymptotically.

Remark 4. If the function $f \in C^v(\Omega)$ is defined on a convex compactly supported domain Ω contained in a ball $O(0,R)$, we can use Hermitian interpolation to extend the function $f(x)$ to whole space that $f \in C^v(I\!R^d)$ and compactly supported on $\overline{O(0,R)}$. Then the quasi- interpolations scheme for the grided data over whole space can be used, that the summation of the quasi- interpolations scheme over the knots outside of the domain Ω depend only on the cross derivatives of the function $f(x)$ on the boundary $\partial\Omega$. Therefor we can easily construct a quasi- interpolation scheme on the compactly supported domain that the scheme possesses a boundary terms, which is derived from the summation over the knots out side of the domain Ω and depend only on the cross derivatives of the function $f(x)$ on the boundary.

5. CONSTRUCTION OF QUASI- INTERPOLANTS FOR MULTIVARIATE SCATTERED DATA

Quasi- interpolation is discussed usually for gridded data or as an operator on a shift- invariant space. It is an interesting problem to generalize quasi- - interpolation to the case of multivariate scattered data. Dyn and Ron [8] and Yoon [27] have derived some schemes by using radial basis functions with scattered centers, the coefficients of the quasi- interpolant are some linear combination of grided data or linear functional of the function $f(x)$. If we can solve the problem of quasi- interpolation for scattered data instead of just allowing scattered centers especially over a compactly supported domain, then such a scheme will be very useful for applications (see de Boor [1]).

Let Ω be a compact convex domain, the scattered set of points $x_j \in \Omega$ be given such that the fill distance or density

$$h = \sup_{x\in\Omega} \inf_{j} \|x - x_j\|$$

is finite and small. The function $f(x) \in C^v(\Omega)$, just follow the Remark 4 in last section we extend the function to whole space by using Hermitian interpolation that $f \in C^v(I\!R^d)$ and compactly supported. Add the grided data points with the spacing h outside of the domain Ω, then we require only to discuss the problem for compactly supported function $f(x)$.

If $\int_{I\!R^d} \Phi(x)dx = 1$, we can adopt the idea of (3.7) to define a quasi-interpolant for scattered data to be

$$I(f) = \sum_j f(x_j)\Phi(\frac{x - x_j}{h^q})\frac{\Delta_j}{h^{qd}}, \quad p + q = 1,$$

where Δ_j are some weights of quadrature. For example, one can take the volume of the a region Ω_k satisfying $x_j \in \Omega_k$, where the $\{\Omega_k\}$ are a partition of $I\!R^d$ such that $\mathrm{Vol}(\Omega_j \cap \Omega_k) = 0, k \neq j$ and $\cup\Omega_k = I\!R^d$ (This quadrature is then the Riemannian summation). A simple choice of Ω_j is via Dirichlet's tessellation:

$$\Omega_j = \{x| \ \|x - x_j\| \leq \|x - x_k\|, \forall k \neq j\}.$$

The discussion in appendix A will not be valid for the case of multivariate scattered data, because good results for numerical integration based on multivariate scattered data are missing. Instead, we use a local approximation $A_j(x)$ introduced by Yoon [27]. The functions $A_j(x)$ are compactly supported on balls $B(x_j, Rh)$ around x_j with radius Rh and satisfy$\sum p(x_j)A_j(x) = p(x)$ for any polynomial $p(x)$ of order at most $u-1$. If we define $\Delta_j = \int A_j(x)dx$, then $\sum \Delta_j f(x_j)$ is a numerical integration scheme of $\int f(x)dx$ with local approximation order u. Since we discuss only the compactly supported function, then the summation is finite and the error of the numerical integration is $\mathcal{O}(h^u)$. For the construction of the approximation $A_j(x)$, moving least square is a good choice, an example is shown in Appendix B.

Remark 5. Theoretically we can use any approximations scheme to get a good quadrature over the domain Ω, however then we will face to solve a large scaled linear system of equation again and lost the advantage of the quasi- interpolation. The reason to use local approximation of $A_j(x)$ is to avoid solving large scaled system of equation (an example is given in the Appendix B by solving some small size of least square problem).

Again, we split the error analysis into

$$\| \sum_j f(x_j)\Phi(\frac{x-x_j}{h^q})\frac{\Delta_j}{h^{qd}} - \int_{I\!R^d} f(y)\Phi(\frac{x-y}{h^q})\frac{dy}{h^{qd}}\|,$$
$$\| \int_{I\!R^d} \hat{f}(w)\hat{\Phi}(h^q w)e^{-ixw}dw - \int_{I\!R^d} \hat{f}(w)e^{-ixw}dw\|,$$

using the identity

$$\int_{I\!R^d} f(y)\Phi(\frac{x-y}{h^q})\frac{dy}{h^{qd}} = \int_{I\!R^d} \hat{f}(w)\hat{\Phi}(h^q w)e^{-ixw}dw,$$

and

$$\int_{I\!R^d} \hat{f}(w)e^{-ixw}dw = f(x).$$

Now we can estimate the error analogously to the discussion of the gridded quasi- interpolant (3.7).

If the numerical integration scheme locally possesses an approximation order of v, the first error behaves like $\mathcal{O}(h^{vp})$, if the function f and its derivatives are bounded.

The second error term depends on the decay of the function $\hat{f}$ and the behavior of $\hat{\Phi}$ near the origin, analogously to the discussion in the last section. We get a bound of the form $\mathcal{O}(h^{\frac{svq}{s+v}})$.

Finally, we can take the optimal $p = \frac{s}{2s+v}$ and $q = \frac{s+v}{2s+v}$, to get a quasi-interpolant with an error estimate $\mathcal{O}(h^{\frac{sv}{2s+v}})$ if $u \geq v$.

THEOREM 3. *Let the kernel $\Phi \in C^u(I\!R^d)$ have an algebraic decay $|\Phi| < o(1+\|x\|)^{-s-d}$ for $x \to \infty$ and satisfy $\int \Phi(x)dx \neq 0$, and $\hat{\Phi} \in C^s$. Assume further $f \in C^v(I\!R^d)$ with $u \geq v$ and $|\hat{f}| < o(1+|t|)^{-d-v}$, a quadrature scheme with weights $\{\Delta_j\}$ that possesses an approximation order v. Then we can put $p = \frac{s}{2s+v}$ and $q = 1-p$ to construct a non- stationary quasi-interpolant with the error bound*

$$|C_\Psi \sum_j f(x_j)\Psi(\frac{x-x_j}{h^q})\frac{\Delta_j}{h^{qd}} - f(x)| \leq \mathcal{O}(h^{\frac{sv}{2s+v}}),$$

where $C_\Psi^{-1} = \int \Psi(x)dx \neq 0$ and Ψ is a linear combination of the scaled shifts of the function Φ. ∎

APPENDIX A: QUADRATURE OVER THE WHOLE SPACE

Here, we will provide some results for numerical integration over the whole space, as required for our study. We stick to simple rules bases on evaluation of points centered in rectangles. By taking suitable linear combinations, one can easily show that gridded integration rules on rectangles based on the midpoint rule, the trapezoidal rule and Simpson's rule will have the same error order. This is a typical fact for integration on the whole space or for periodic functions, and there will not be a substantial improvement by taking more complicated rules.

THEOREM 4. *If $\Phi \in C^u(I\!R^d)$ and compactly supported on a ball around the origin with radius R, then*

$$\left| \int_{I\!R^d} \Phi(x-y)dx - h^d \sum_{j\in Z\!\!Z^d} \Phi(jh-y) \right| \le CR^d \|\Phi^{(u)}\|_\infty h^u$$

uniformly, independent of y.

Proof: Take locally polynomial interpolation over the every interval $[juh,(j+1)uh]$ to get

$$\begin{aligned} &|\textstyle\int_{I\!R^d} \Phi(x-y)dx - h^d \sum_{j\in Z\!\!Z^d} \sum_{|i|_\infty \le u} w_i \Phi(juh+ih-y)| \\ \le\ & CR^d \|\Phi^{(u)}\|_\infty h^u \end{aligned} \tag{A.14}$$

where $i,j \in Z\!\!Z^d$. Then shifts of the formula (A.14) and the average of the shifts of the formula (A.14), equivalently the formula with rectangular rule

$$\begin{aligned} &\frac{h^d}{u^d} \sum_{|l|<u} \sum_{j\in Z\!\!Z^d} \sum_{|i|\le u} w_i \Phi(juh+ih+lh-y) \\ &= h^d \sum_{j\in Z\!\!Z^d} \Phi(jh-y) \end{aligned} \tag{A.15}$$

possess the same approximation order as (A.14). ∎

For the function $|\Phi(x)| < o(1+|x|)^{-s-d}$, we can split the integral and the summation into two parts with $|x| < h^{-r}$ and $|x| > h^{-r}$. Then the summation and the integral for $|x| > h^{-r}$ are bounded by $\mathcal{O}(h^{sr})$. Furthermore, the difference of the summation and the integral for $|x| < h^{-r}$ are bounded by $\mathcal{O}(h^{u-rd})$ by using Theorem 4. The best error estimate can be obtained by setting $sr = u - rd$. This yields

THEOREM 5. *If $\Phi \in C^u$ and $|\Phi(x)| < o(1+|x|)^{-s-d}$, then*

$$|\int_{I\!R^d} \Phi(x-y)dx - h^d \sum_{j\in Z\!\!Z^d} \Phi(jh-y)| \le \mathcal{O}(h^{\frac{su}{s+d}}).$$

uniformly, independent of y. ∎

APPENDIX B: LOCAL POLYNOMIAL REPRODUCTION APPROXIMATION AND THE QUADRATURE OVER WHOLE SPACE FOR SCATTERED DATA

We would like to construct multivariate functions $A_j(x)$ that are compactly supported on $O(x_j, Rh)$ and satisfy $\sum A_j(x)p(x_j) = p(x)$ for any polynomial $p(x)$ whose degree is less than or equal to u. The only reference for an explicit construction seems to be Sibson's interpolation [21] with $u = 1$. Thus we give an example of general construction here. The approach is in fact the moving least square.

For simplicity we assume scattered data points x_j with density h and take a basis $p_1(x), \ldots, p_q(x)$ of multivariate polynomials with degree less than or equal to u. For every j we take a subset Y_j of the points such that

$$Y_j = \{y_{j,k}\} \subset \{x_k\} \cap O(x_j, Rh),$$

where $R > 3u$. Then for any x_j the matrices $P_{x_j} = (p_\ell(y_{j,k}))_{\ell,k}$ are all of full rank (see e.g. [24]). Let f_{x_j} be vectors with the entries $f(y_{j,k})$. We construct a least squares approximation for every j by polynomials using the data near the point x_j by

$$T_j(x) = (p_1(x), \ldots, p_q(x))(P_{x_j}^T P_{x_j})^{-1} P_{x_j}^T f_{x_j}.$$

Furthermore we define

$$w(x,y) = \begin{cases} \dfrac{\|x-y\|^2}{(Rh)^2 - \|x-y\|^2} & \|x-y\|^2 \le (Rh)^2 \\ \infty & \|x-y\|^2 > (Rh)^2 \end{cases}$$

and construct the generalized Shepard's interpolation

$$f^*(x) = \sum T_j(x) L_j(x), \tag{B.16}$$

where

$$L_j(x) = (\frac{1}{w^{u+1}(x, x_j)})/(\sum_{\|x_k - x\| < Rh} \frac{1}{w^{u+1}(x, x_k)}).$$

If the data comes from a polynomial $T(x)$, the least squares approximation is polynomial reproducing that satisfies $T_j(x) = T(x)$ and then $f^*(x) = T(x)$ from the properties of Shepard's interpolation. The scheme (B.16) is a linear combination of f_j, so that

$$f^*(x) = \sum f_j A_j(x),$$

and if $f_j = T(x_j)$ we have

$$T(x) = \sum T(x_j) A_j(x),$$

where $A_j(x)$ are composed of some terms with factors $L_k(x)$ satisfying $|x_j - x_k| \le Rh$. Thus the $A_j(x)$ are compactly supported on $B(x_j, 2Rh)$ and reproducing polynomials up to degree u.

Now we construct the quadrature scheme over the compact domain Ω. Let $\Delta_j = \int_\Omega A_j(x)dx$, if the function f is compactly supported on Ω then

$$\begin{aligned}&\sum f(x_j)\Delta_j = \int_\Omega \sum f(x_j)A_j(x)dx = \int_\Omega f(x) + \mathcal{O}(h^u)dx\\ &= \int_\Omega f(x)dx + \mathcal{O}(h^u) = \int_{I\!R^d} f(x)dx + \mathcal{O}(h^u).\end{aligned}$$

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