Basis Functions for Rational Continuity

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Abstract: The parametric or geometric continuity of a rational polynomial curve has often been obtained by requiring the homogeneous polynomial curve associated with the rational curve to possess parametric or geometric continuity, respectively. Recently this approach has been shown overly restrictive. We make use of the necessary and sufficient conditions of rational parametric continuity for defining basis functions for the homogeneous representation of a rational curve.

These functions are represented in terms of shape parameters of rational continuity, which are introduced due to these exact conditions. The shape parameters may be varied globally, affecting the entire curve, or modified locally thereby affecting only a few segments. Moreover, the local parameters can be represented as continuous or discrete functions. Based on these properties, we introduce three classes of basis functions which can be used for the homogeneous representation of rational parametric curves.

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1 Introduction

The rational formulation of polynomials and splines has received considerable interest in the areas of computer graphics and geometric modeling. The main advantage of the rational form is its ability to represent conic curves and quadric surfaces, as well as free-form curves and surfaces (Piegl 1985; Piegl 1986a; Piegl 1986b; Piegl 1986c; Piegl and Tiller 1987; Salmon 1879; Tiller 1983; Versprille 1975). Moreover, it is invariant under affine as well as projective transformations, though the latter is possible with changed weights (Lee 1987).

A single rational polynomial usually does not have enough freedom to represent a given curve; several rational polynomial segments are used instead. To obtain a curve of satisfactory smoothness, the segments must connect with some amount of continuity. Thus, the use of rational curves, independent of the particular variety, creates the problem of connecting rational polynomial segments to form piecewise rational curves that are smooth.

To obtain rational curves with either parametric or geometric continuity, the parametric continuity constraints or geometric continuity constraints, respectively, have been applied to the components of the curve in homogeneous coordinates, but not to the components of the rational curve (Barsky 1988a; Boehm 1987; Farin 1983; Goldman and Barsky 1989a; Goldman and Barsky 1989b; Joe 1989; Tiller 1983). If the homogeneous curve satisfies the relevant continuity constraints then the rational curve will have the corresponding continuity.

There are two kinds of continuity for parametric curves: *parametric* continuity and *geometric* (or *visual*) continuity. A curve is said to possess parametric continuity (denoted C^n) if each segment of the curve is C^n and the adjacent segments are connected with C^n continuity at the joints. There are two notions of geometric continuity. The first is based on parametric continuity after a suitable reparametrization. That is, a curve is said to be geometrically continuous (denoted G^n) if there exists some *reparametrization* of its segments such that the resulting curve is C^n. The reparametrization criterion on its segments leads to the derivation of *Beta-constraints* (Barsky 1988b; Barsky and DeRose 1989; Barsky and DeRose 1990; DeRose 1985). The second notion of geometric continuity of parametric curves is based on the continuity of *Frenet Frame* and higher order curvatures (Boehm 1985; Boehm 1987; Dyn and Micchelli 1985; Hagen 1985). A curve possesses geometric continuity G^n if each of its components satisfies the corresponding constraints at the joints.

However, these constraints are sufficient but not necessary for the continuity of rational curves (Hohmeyer and Barsky 1989). The necessary and sufficient constraints for rational continuity (parametric or geometric continuity), as applied to the homogeneous curve, give rise to *shape parameters*. These shape parameters are different from and in addition to those obtained from the geometric continuity of polynomial or rational curves and surfaces (dubbed as *Betas*). They are used for modifying the curve independent of the control vertices. Experience has shown us that shape parameters provide a designer with intuitive control of shape. Some of the properties of these parameters are very useful for the

development of modern geometric modeling system.

In this paper, we introduce three categories of *basis functions*[3] for the homogeneous representation of rational curves. These basis functions are based upon the necessary and sufficient conditions for *rational parametric continuity*. The different categories are obtained by varying the shape parameters globally, which affects the entire curve, or locally, which affects only a few segments only. For the local variation, the shape parameters may be either continuously varying functions or be discretely specified at the knot values. The resulting basis functions differ in the amount of local control (with respect to shape parameters) and evaluation cost.

The rest of the paper is organized in the following manner: In Section 2, we give a brief overview of rational curves and specify the notation. The necessary and sufficient conditions for the parametric continuity of rational curves are mentioned in Section 3. They are used in Section 4 for defining the basis functions for the homogeneous representation of rational curves with uniform shape parameters. In Section 5, the uniform shape parameters are generalized into continuously varying shape parameters by quintic Hermite interpolation. Section 6 specifies a discretely-shaped homogeneous basis. Finally, in Section 7 we present the formulation which can be used for constructing rational curves satisfying the necessary and sufficient conditions for geometric continuity.

2 Rational Curves

A *rational polynomial function* is a scalar function, $r : \mathcal{R} \rightarrow \mathcal{R}$ that can be expressed as

$$r(u) = \frac{f(u)}{g(u)}$$

where *f(u)* and *g(u)* are polynomials in *u*. We will restrict our use of the word polynomial in the following manner: a rational function is not a polynomial function unless its denominator divides its numerator. Such a function is at times referred to as an *integral* function to distinguish it from a rational function. A *curve* is simply a vector-valued function $\mathbf{q} : \mathcal{R} \rightarrow \mathcal{R}^d$. A *rational curve* is a vector-valued function, each component of which is a rational function. There are at least two ways to represent a rational curve (Hohmeyer and Barsky 1989). First, the curve $\mathbf{q}(\mathrm{u})$ can be thought of as a vector-valued function, each component of which is a rational function. Alternatively, $\mathbf{q}(\mathrm{u})$ can be thought of as a vector-valued function $\mathbf{Q} : \mathcal{R} \rightarrow \mathcal{R}^{d+1}$ with a projection function that projects $(x_1, \ldots, x_d, x_{d+1})$ to $(x_1/x_{d+1}, \ldots, x_d/x_{d+1})$. That is, the function is originally in $(d + 1)$-dimensional space but is then projected down to a *d*-dimensional space.

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[3]Any homogeneous representation of a fixed degree will be a linear combination of these functions. However, they do not form any basis spanning a vector space, since the rational curves of a fixed degree do not constitute a vector space.

In the first representation, we write:

$$\mathbf{q}(u) = (r_1(u), r_2(u), \ldots, r_d(u))$$

where each component function, $r_i(u)$, is a rational function of the form $\frac{f_i(u)}{g(u)}$. In the second representation, one might consider the same curve as a polynomial curve whose range is the homogeneous coordinate system of dimension $d+1$. The rational curve $\mathbf{q}(u)$, discussed above, would be represented in this scheme by the polynomial curve $\mathbf{Q}(u)$, where

$$\mathbf{Q}(u) = (f_1(u), \ldots, f_d(u), g(u))$$

which is in *(d + 1)*-dimensional space.

We refer to $\mathbf{Q}(u)$ as the *homogeneous curve* associated with $\mathbf{q}(u)$, the *rational curve* or the *projected curve*, and $\mathbf{q}(u)$ as the *projection* of $\mathbf{Q}(u)$. The homogeneous curve is not unique, whereas the projection is. Moreover, we use italics to indicate scalar-valued functions, such as $f(u)$ or $g(u)$, boldface lower case to indicate vector-valued rational curves, such as $\mathbf{q}(u)$, whose range is $\mathcal{R}^d$, and bold face upper case to indicate the associated homogeneous polynomial curve, such as $\mathbf{Q}(u)$, whose range is $\mathcal{R}^{d+1}$. The *degree* of a rational curve is the degree of its homogeneous representation.

To illustrate more concretely, consider a curve formulation such as the rational Bézier curve, the rational B-spline curve, or the rational Beta-spline curve of a fixed degree. Each is a function $\mathbf{q} : \mathcal{R} \to \mathcal{R}^{\mathbf{d}}$ that can be expressed as

$$\mathbf{q}(u) \quad = \quad \sum_{i=0}^{m} \mathbf{V}_i \left[\frac{w_i B_i(u)}{\sum_{j=0}^{m} w_j B_j(u)} \right] \tag{1}$$

where $\mathbf{V}_i \in \mathcal{R}^d$ are the control vertices, $B_i(u)$ are the basis functions for the homogeneous curve, defined as a polynomial curve, and w_i are the weights, for $i = 0, \ldots, m$, which are independent of the control vertices and the basis functions. One can also consider such a curve as a polynomial Bézier curve, B-spline curve, or Beta-spline curve in the homogeneous space:

$$\mathbf{Q}(u) = \sum_{i=0}^{m} \mathbf{W}_i B_i(u)$$

where $\mathbf{W}_i$ are control vertices in $\mathcal{R}^{d+1}$, whose coordinates are expressed as $\mathbf{W}_i = (w_i \mathbf{V}_i, w_i)$ for $i = 0, \ldots, m$. Whenever the $\mathcal{R}^d$ coordinates of $\mathbf{q}(\mathrm{u})$ are required, the division must be performed. For example, the circle in Fig. I is a rational curve in $\mathcal{R}^2$, which is a projection of a polynomial curve in $\mathcal{R}^{2+1}$. The points on the $\mathcal{R}^2$ curve are obtained by projecting the $\mathcal{R}^{2+1}$ curve onto the $w = 1$ plane.

The advantage of this perspective is that the algorithms to manipulate rational curves (i.e. evaluation, subdivision, degree elevation etc.) can be obtained by using the corresponding algorithm for polynomial curves. This method of reducing a problem associated with a rational curve to the analogous problem for its homogeneous counterpart has also

been used for the problem of continuity, parametric or geometric, between rational segments. It is true that if the homogeneous curve satisfies the relevant continuity constraints then the rational curve will have the corresponding continuity. However, there are cases when the projected curve satisfies the required continuity constraints, whereas the corresponding homogeneous curve does not (Hohmeyer and Barsky 1989). Thus, the above approach is overly restrictive. One such case is shown in Fig. I. A circle is being generated with C^0 rational curves. In the figure, the circle has been shown as a projection of a homogeneous curve, $\mathbf{Q}(u) \in \mathcal{R}^{2+1}$, onto the $w = 1$ plane. The axis of the cone (whose base is also shown) lies along the w-axis. The tip is at the origin and the homogeneous curve $\mathbf{Q}(u)$ consists of four segments, lying on the surface (including the base) of the cone and it is C^0 continuous. $\mathbf{Q}(u)$ is neither C^1 nor even G^1, whereas $\mathbf{q}(u)$ is trivially C^2.

The necessary and sufficient constraints on the homogeneous curve engenders shape parameters (Hohmeyer and Barsky 1989). The basis functions for the homogeneous representation, $B_i(u)$, as specified in (1), become functions of these shape parameters. Changing the shape parameters changes the projected curve, while still satisfying the necessary continuity constraints.
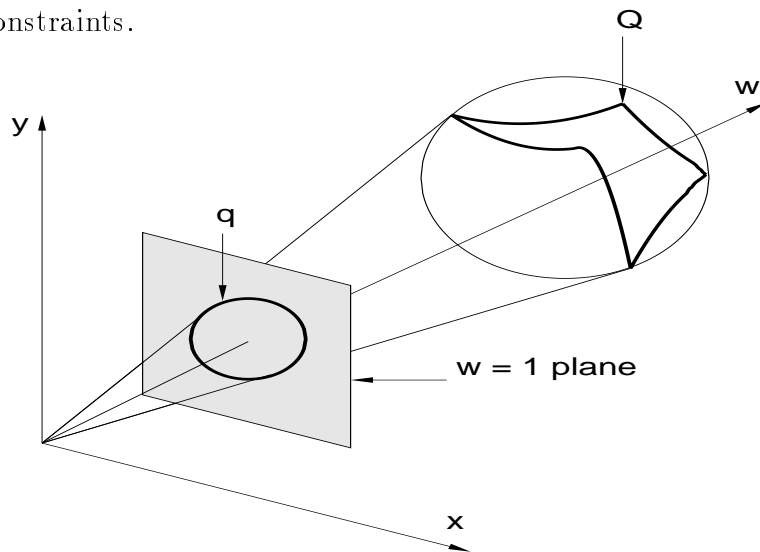

Fig. I
A circle $\mathbf{q}(u)$ represented by a piecewise rational $\mathbf{Q}(u)$.
There are *kinks* in $\mathbf{Q}(u)$, but $\mathbf{q}(u)$ is C^2 continuous.

3   Continuity Constraints

In our case the *smoothness* of a curve has been measured by testing the parametric continuity of the functions defining the curve. If we have a homogeneous curve $\mathbf{Q}(u)$ that is C^n, then the projection $\mathbf{q}(u)$ of $\mathbf{Q}(u)$ will also be C^n. However, the converse is not true. As mentioned earlier, there are homogeneous curves $\mathbf{Q}(u)$ that are not C^n even though

their projections $\mathbf{q}(u)$ are C^n. The curve $\mathbf{q}(u)$ is C^n, say at $u = u_0$, if and only if each component of $\mathbf{q}$(u), which is of the form $f_i(u)/g(u)$, is C^n at $u = u_0$. Consider a generic quotient function of the type $f(u)/g(u)$. The necessary and sufficient conditions so that such a quotient function would be C^n at a point $u = u_0$ are that there exist α_i's such that

$$\lim_{u \to u_0+} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \\ \vdots \\ f^{(n)}(u) \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & 0 & \ldots & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 & \ldots & 0 & 0 \\ \alpha_2 & 2\alpha_1 & \alpha_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \alpha_n & \binom{n}{1}\alpha_{n-1} & \binom{n}{2}\alpha_{n-2} & \ldots & \binom{n}{n-1}\alpha_1 & \alpha_0 \end{bmatrix} \lim_{u \to u_0-} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \\ \vdots \\ f^{(n)}(u) \end{bmatrix} \quad (2)$$

and similarly for the function $g(u)$ (with the same set of α_i's). At any point $u = u_0$, all the functions $f_i(u)$ and g(u) need to satisfy the above relation for any set of α's where $\alpha_0 \neq 0$ (Hohmeyer and Barsky 1989). In other words, each component of $\mathbf{Q}$(u) is related by the above constraint at each knot value and the set of α's must be the same for all components.

We use the α's defined above as shape parameters. Changing the values of the α's gives a different projected curve $\mathbf{q}$(u), which maintains the desired continuity. In most computer graphics applications, cubic curves are used because they provide a balance between computational cost and the desired smoothness and flexibility. The desired smoothness is frequently ensured by requiring the curve to be C^2 continuous in the desired interval. So we formulate a cubic basis for the homogeneous curve and generate a C^2 projected curve $\mathbf{q}$(u). Thus, there are *three shape parameters*, α_0, α_1, and α_2, that determine our projected curve $\mathbf{q}$(u). For this case, (2) can be rewritten in the following manner (n=2):

$$\lim_{u \to u_0+} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & 2\alpha_1 & \alpha_0 \end{bmatrix} \lim_{u \to u_0-} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix} \quad (3)$$

We will initially use α's as *uniform* shape parameters. That is, each shape parameter has the same value at the knots. We later generalize them to *continuous* functions of shape parameters, where the user is not constrained to use the same value and it offers maximum local control of these parameters. The latter formulation turns out to be computationally expensive. Therefore, at the end we introduce a discretely-shaped basis for the homogeneous curve, which provides a balance between evaluation cost and the degree of local control of shape parameters. For each case, we will define a basis function and use (1) to determine the projected curve $\mathbf{q}$(u).

4   Uniform Shape Parameters

In this section, we define the basis functions for the homogeneous curve. Our terminology is similar to that described in (Bartels et. al. 1987, Chapter 2). The rational curve is

obtained by projecting the linear combinations of the basis functions. The scalar multiples used for the linear combinations are determined by the control vertices and weights. The representation of a rational curve $\mathbf{q}(u)$, as defined in (1), leads to the formation of the following functions:

$$R_i(u) = \frac{w_i B_i(u)}{\sum_{j=0}^{m} w_j B_j(u)}, \quad i = 0, \ldots, m \tag{4}$$

where the $B_j(u)$ are the basis functions for the homogeneous curve. The resulting curve $\mathbf{q}(u)$ must be C^2 continuous in the specified interval. The desired continuity could be achieved if $R_i(u)$ were themselves C^2 functions, since the rational curve is a linear combination of these functions. This happens if each $B_j(u)$ and its first two derivatives are related by (3) at u_i. To simplify, we work over a *uniform knot* sequence and derive the canonical basis function in a manner similar to that described in (Bartels et. al. 1987).

4.1   The Basis Functions

Each $B_i(u)$ consists of four basis segments. We denote these segments by $b_i(u)$, $i = 0, 1, 2, 3$. Each segment is a cubic polynomial of the form

$$b_i(u) = p_i + q_i u + r_i u^2 + s_i u^3, \quad u \in [0, 1) \quad i = 0, 1, 2, 3$$

At the knot values, the $B_j(u)$ are related by (3). This engenders the following fifteen relations among the $b_i(u)$:

$$
\begin{array}{rclrclrcl}
0 &=& b_0(0), & 0 &=& b_0^1(0), & 0 &=& b_0^2(0) \\
\alpha_0 b_0(1) &=& b_1(0), & \alpha_1 b_0(1) + \alpha_0 b_0^1(1) &=& b_1^1(0), & \alpha_2 b_0(1) + 2\alpha_1 b_0^1(1) + \alpha_0 b_0^2(1) &=& b_1^2(0) \\
\alpha_0 b_1(1) &=& b_2(0), & \alpha_1 b_1(1) + \alpha_0 b_1^1(1) &=& b_2^1(0), & \alpha_2 b_1(1) + 2\alpha_1 b_1^1(1) + \alpha_0 b_1^2(1) &=& b_2^2(0) \\
\alpha_0 b_2(1) &=& b_3(0), & \alpha_1 b_2(1) + \alpha_0 b_2^1(1) &=& b_3^1(0), & \alpha_2 b_2(1) + 2\alpha_1 b_2^1(1) + \alpha_0 b_2^2(1) &=& b_3^2(0) \\
\alpha_0 b_3(1) &=& 0, & \alpha_1 b_3(1) + \alpha_0 b_3^1(1) &=& 0, & \alpha_2 b_3(1) + 2\alpha_1 b_3^1(1) + \alpha_0 b_3^2(1) &=& 0
\end{array}
$$

There are four cubic functions, $b_0(u), b_1(u), b_2(u)$ and $b_3(u)$, which constitute $B_j(u)$. Each has four unknowns (equal to the order of each polynomial). Thus, there are sixteen variables in all. Any sixteen independent variables, each being defined over the real field $\mathcal{R}$, belong to the 16-dimensional *vector space* $\mathcal{R}^{16}$. In our case, the variables are not independent. On applying the fifteen constraints we obtain a 1-dimensional *subspace* of the 16-dimensional vector space. Geometrically, the fifteen constraints given above determine a line in the 16-dimensional space. Moreover, that line passes through the origin (as it is a vector subspace). Any point lying on that line, except the origin, would determine a solution set to the sixteen variables. That solution set is of the following form:

$$[s_1 g, s_2 g, \ldots, s_{15} g, s_{16} g]^T$$

where each s_i is a function of α_0, α_1, and α_2.

The variable g acts as a *normalizing factor*. If $g = 0$ then the basis function $B_j(u)$ becomes equal to a zero function (it is zero throughout). We let $b_i(u)$ be defined in terms of g. The rational basis functions $R_i(u)$ are independent of g, as it cancels out in the numerator and denominator terms of (4), since $g \neq 0$.

In the case of polynomial B-splines and Beta-splines, formed from C^2 and G^2 continuity constraints, respectively, the fact that the basis functions should form a *partition of unity* gives rise to the sixteenth constraint and the system of equations used for determining the coefficients of basis functions has a unique solution (Bartels et. al. 1987). The fact that $R_i(u)$, $i = 0, \ldots, m$, should form a partition of unity is embedded in their formulation and no such constraint is required on $B_i(u)$. That property engenders another shape parameter, expressed above as g. If we use a different value of g for each $B_i(u)$, then it has a local effect on the curve. The effect is similar to that of the weights for rational curves expressed as (1). However, the weights are present due to the homogeneous representation of the vertices; i.e., $\mathbf{W}_i = (w_i \mathbf{V}_i, w_i)$. If we want to use these basis functions for homogeneous representations of tensor product surfaces, the g's will affect the surface in a different manner as compared to the weights.

Using Macsyma (Fateman 1982), we obtain the following basis functions[4] for the homogeneous representation of the rational curve:

$$
\begin{aligned}
b_0(u) &= gu^3 \\
b_1(u) &= g\,(\alpha_0 + [\alpha_1 + 3\alpha_0]u + [\frac{\alpha_2 + 6\alpha_1 + 6\alpha_0}{2}]u^2 - [\frac{\alpha_2 + 6\alpha_1 + 6\alpha_0}{2}]u^3) \\
b_2(u) &= g\,([\alpha_0\alpha_1 + 4\alpha_0^2] + [\frac{\alpha_0(4\alpha_1 - \alpha_2) + 2\alpha_1^2}{2}]u + \\
&\quad [\alpha_0(\alpha_2 - 6\alpha_1) - 2\alpha_1^2 - 6\alpha_0^2]u^2 + [\frac{\alpha_0(6\alpha_1 - \alpha_2) + 2\alpha_1^2 + 6\alpha_0^2}{2}u^3]) \\
b_3(u) &= g\alpha_0^3(1 - 3u + 3u^2 - u^3)
\end{aligned}
\tag{5}
$$

If we set

$$ \alpha_0 = 1, \qquad \alpha_1 = 0, \qquad \alpha_2 = 0 $$

then the above basis reduces to a basis for the uniform cubic B-spline (Bartels et. al. 1987). These are the default values of the shape parameters and the resulting function is a basis for the homogeneous representation of a cubic rational B-spline with uniform knot spacing. To determine a curve, we select a set of control vertices $\mathbf{V}_i$ and use (1) to define the curve. The $B_i(u)$ used in the definition of $R_i(u)$ in (4) are defined in terms of the segments $b_i(u)$ (5). Each $B_i(u)$ is nonzero over four successive intervals only, say from u_i to u_{i+4}. Thus, over the knot interval $[u_{i+3}, u_{i+4})$, the portion of the curve defining $\mathbf{q}(u)$ can be represented as

$$
\mathbf{S}_i(u) = \frac{\mathbf{V}_i w_i b_3(u) + \mathbf{V}_{i+1} w_{i+1} b_2(u) + \mathbf{V}_{i+2} w_{i+2} b_1(u) + \mathbf{V}_{i+3} w_{i+3} b_0(u)}{w_i b_3(u) + w_{i+1} b_2(u) + w_{i+2} b_1(u) + w_{i+3} b_0(u)}
\tag{6}
$$

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[4]This is one of the many of formulations of the basis functions. We can let g be a function of the α's. We tried many combinations and this is the simplest we could achieve in terms of the size of the coefficients.

The whole curve $\mathbf{q}(\mathrm{u})$ is composed of $m - 2$ segments, $\mathbf{S}_i(u)$, $i = 0, \ldots, m - 3$.

It is also desirable for the resulting curve $\mathbf{q}(\mathrm{u})$ obtained from (1) to lie in the convex hull of the control vertices $\mathbf{V}_i$. Using (4), $\mathbf{q}(\mathrm{u})$ can be expressed as

$$\mathbf{q}(u) = \sum_{i=0}^{m} \mathbf{V}_i R_i(u). \tag{7}$$

Since we are considering a uniform knot sequence, all these relations are defined for $u \in [0, 1)$. From the definition of $R_i(u)$ in (4), we know that:

$$\sum_{i=0}^{m} R_i(u) = 1$$

The only other condition that needs to be satisfied for the convex hull property is

$$R_i(u) \geq 0, \qquad u \in [0, 1), \qquad i = 0, \ldots, m \tag{8}$$

A sufficient condition for the above inequality may be obtained by non-negative values of α_0, α_1, and α_2. Large positive values of α_2 result in the loss of convex hull property. Each segment of $\mathbf{q}(\mathrm{u})$ lies in the convex hull of the four vertices used for defining it (6).

4.2 Varying the α's

Varying the α's changes the rational curve. A detailed analysis and explanation of their behavior can be found in (Manocha and Barsky 1990). The shape parameters α_0 and α_1 are found to behave very similar to the *bias* and *tension* parameters, β_1 and β_2, respectively, of the polynomial and rational Beta-spline curves (Barsky 1988b; Barsky and Beatty 1983; Barsky and DeRose 1989; Barsky and DeRose 1990; DeRose 1985). The planar rational curves resulting from different values of α_0 and α_1 have been shown in Fig. II and Fig. III respectively. Some negative values lead to the occurrence of loops or self-intersections. The positions where adjacent segments meet, called *joints*, of the rational curve are invariant with respect to altering α_2 (as shown in Fig. IV).
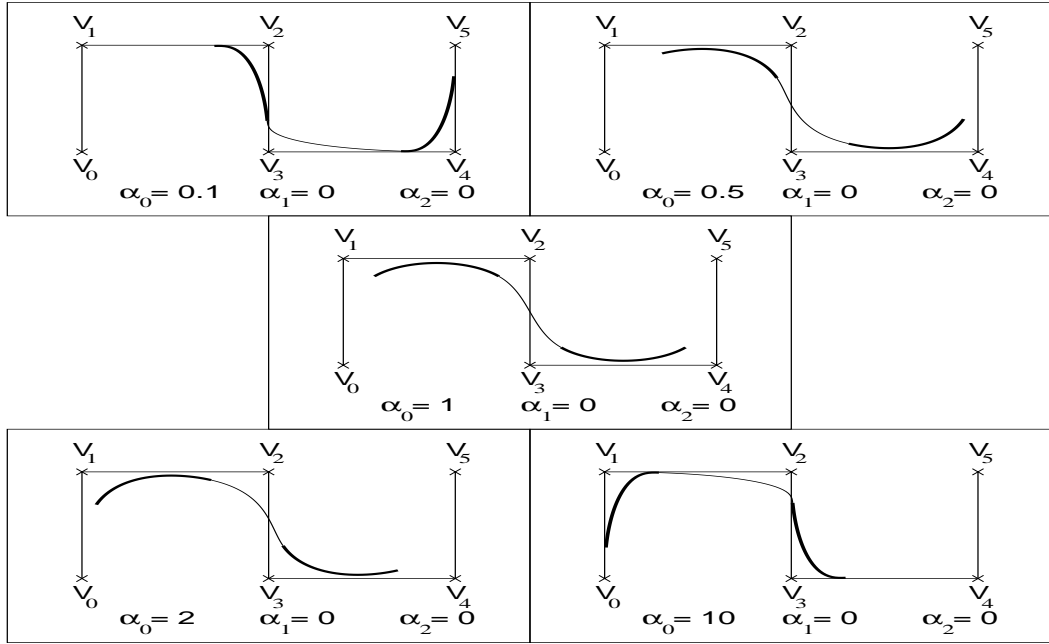
Fig. II
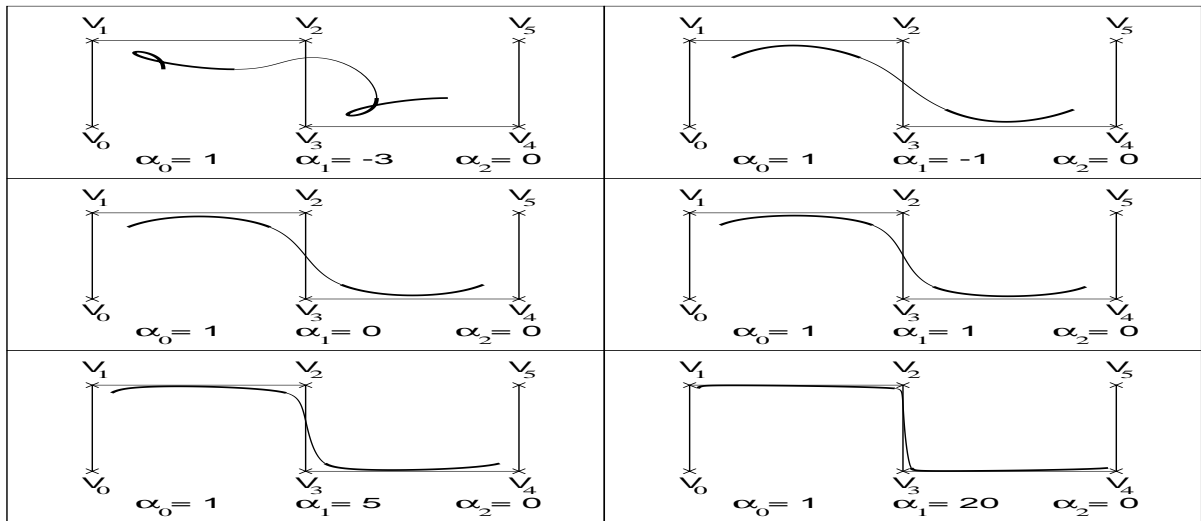The shape of the rational curves corresponding to different values of α_0.

Fig. III
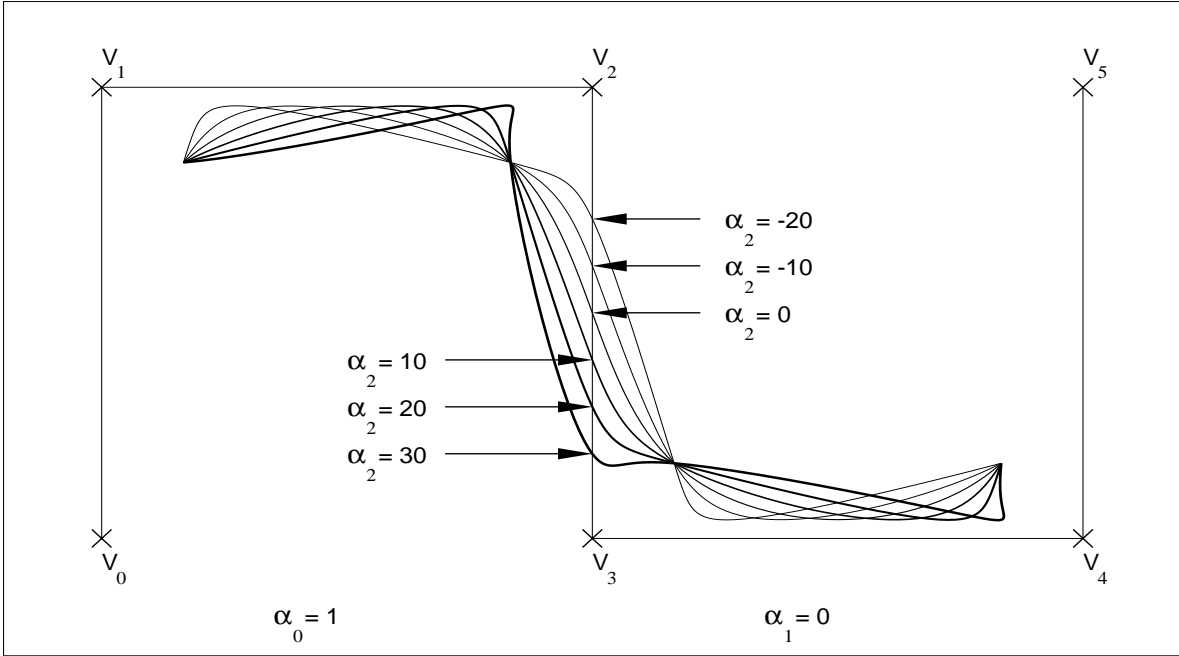Different rational curves corresponding to different values of α_1.

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Fig. IV

The shape of the rational curves corresponding to different values of α_2.

5   Continuous Shape Parameters

In the previous sections the formation and usage of *basis functions*, have been based on the fact that the α's are *uniform* shape parameters; i.e., each parameter assumes a unique value. In this section, they are generalized to *continuous* shape parameters (though we address them as continuous functions of shape parameters, hereafter) each varying continuously along the curve. The continuous analogues of α_0, α_1, and α_2 will be denoted by $\alpha_{0_i}(u)$, $\alpha_{1_i}(u)$, and $\alpha_{2_i}(u)$, respectively, and describe the value of each shape parameter along the curve segment $\mathbf{SS}_i(u), i = 0, \ldots, m$. $\mathbf{SS}_i(u)$ represents a segment of the rational curve based on continuous function of shape parameters. Its formulation is similar to $\mathbf{S}_i(u)$ in (6), except that $b_i(u)$ is replaced by $bb_i(u)$ (described below). This generalization enables the user to have more precise control over the shape of the curve. The user is no longer constrained to choose a unique value for each shape parameter over the entire curve. The different values of the shape parameters can be used to reflect the local character of the shape parameters along the curve. This is in addition to local control, which is available with respect to the control vertices.

For simplicity, we again choose to use a uniform knot sequence ($u_{i+1} = u_i + 1$) to show the derivation of continuous functions of shape parameters. Let γ_{0_i}, γ_{1_i} and γ_{2_i} be the values that are associated with the continuous functions of shape parameters corresponding to the

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knot value u_i (as shown in fig. V). They are specified by the user and used for changing the curve. We will refer to these values as the *user_specified* values for the continuous functions of shape parameters. Now each component of the homogeneous curve, $\mathbf{Q}(u)$, is modified so that in the neighborhood of the knot u_i, it is related in the following manner:

$$\lim_{u \to u_i+} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix} = \begin{bmatrix} \gamma_{0_i} & 0 & 0 \\ \gamma_{1_i} & \gamma_{0_i} & 0 \\ \gamma_{2_i} & 2\gamma_{1_i} & \gamma_{0_i} \end{bmatrix} \lim_{u \to u_i-} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix} \tag{9}$$

The above relation is satisfied by each component of $\mathbf{Q}(u)$, so that the projected curve, $\mathbf{q}(u)$, is C^2 continuous. We choose the basis functions for the $f_i(u)$'s and $g(u)$ (components of $\mathbf{Q}(u)$) by replacing the uniform parameters by their continuous analogues in (5)

$$bb_0(u) = gu^3$$

$$bb_1(u) = g(\alpha_{0_i}(u) + [\alpha_{1_i}(u) + 3\alpha_{0_i}(u)]u + \frac{[\alpha_{2_i}(u) + 6\alpha_{1_i}(u) + 6\alpha_{0_i}(u)]}{2}u^2 -$$
$$\frac{[\alpha_{2_i}(u) + 6\alpha_{1_i}(u) + 6\alpha_{0_i}(u)]}{2}u^3) \tag{10}$$

$$bb_2(u) = g([\alpha_{0_i}(u)\alpha_{1_i}(u) + 4(\alpha_{0_i}(u))^2] + \frac{[\alpha_{0_i}(u)(4\alpha_{1_i}(u) - \alpha_{2_i}(u)) + 2(\alpha_{1_i}(u))^2]}{2}u +$$
$$[\alpha_{0_i}(u)(\alpha_{2_i}(u) - 6\alpha_{1_i}(u)) - 2\alpha_{1_i}(u)^2 - 6(\alpha_{0_i}(u))^2]u^2 +$$
$$\frac{[\alpha_{0_i}(u)(6\alpha_{1_i}(u) - \alpha_{2_i}(u)) + 2(\alpha_{1_i}(u))^2 + 6(\alpha_{0_i}(u))^2]}{2}u^3)$$

$$bb_3(u) = g(\alpha_{0_i}(u))^3(1 - 3u + 3u^2 - u^3)$$

The basis functions of $\mathbf{Q}(u)$ can be represented by many formulations. It is certainly of interest to know about the class of functions which can be used for the continuously varying shape parameters so that (9) holds. Many factors play an important role in the generation of the family of functions. Frequently (i.e., for the choice of *user_specified* values and continuous functions for the shape parameters), the convex hull property would not be retained and at times visually undesirable effects like cusps or loops are introduced in the curve. We think that using functions, which represent $\alpha_{j_i}(u)$ as piecewise and local in nature, as compared to global functions, would be of great use since they represent the local behavior of these rational curves in the best possible manner, thereby providing us a lot of flexibility for our applications.

Each of the $\alpha_{j_i}(u)$, $(j \in \{0, 1, 2\})$, is a piecewise function that must interpolate the *user_specified* values at the knot value. The fact that the resulting function must satisfy the constraints mentioned in (9) limits the class of functions that can be used for $\alpha_{j_i}(u)$. Currently, our aim is to choose one of minimal degree, in terms of u, since it will be more efficient to evaluate. We determine the continuous functions of shape parameters as polynomials of degree five. These functions are obtained by a special case of quintic interpolation. This is derived in detail in (Manocha and Barsky 1990) yielding:

$$\alpha_{j_i}(u) = \gamma_{j_{i-1}} + (\gamma_{j_i} - \gamma_{j_{i-1}})u^3[10 - 15u + 6u^2]$$
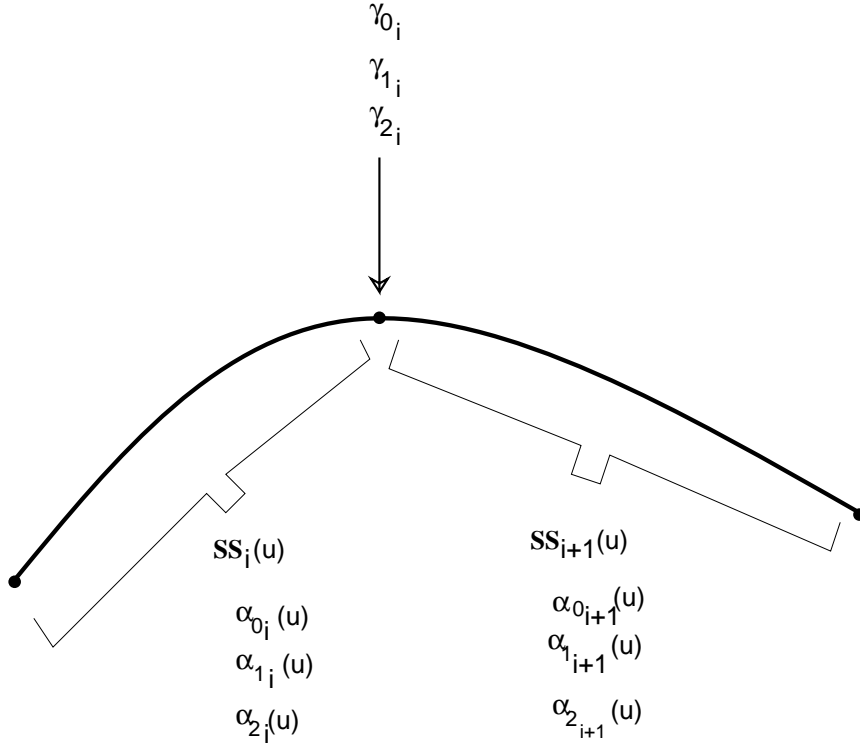
11

Fig. V

Continuous and discrete shape parameters for curves

5.1 Locality

We have derived the basis functions for the homogeneous curve by using continuous functions of shape parameters. This representation gives the user more precise control over the shape of the curve. Moreover, a change in any γ_{j_i} affects only two adjacent segments $\mathbf{SS}_i(u)$ and $\mathbf{SS}_{i+1}(u)$ as their shape parameter functions are affected by it. However, we pay a price for this extra control in terms of cost of evaluation. Our basis functions are now polynomials of degree eighteen. But they can be decomposed into products of polynomials of degree five and three.

Any movement of a control vertex affects only four segments. This choice is independent of the choice of shape parameters (whether uniform or continuous). This occurs because any basis function will be nonzero only over four intervals.

5.2 Varying the γ's

Different values of γ_{j_i} affect the adjacent segments. A detailed analysis of the behavior of the rational curves in terms of these parameters is given in (Manocha and Barsky 1990). The parameters γ_0 and γ_1 behave very similarly to the *bias* and *tension* parameters of the

polynomial and rational Beta-spline curves (Bartels et. al. 1987; Barsky 1988). In Fig. VI, the planar rational curves resulting from different values of γ_0's are shown. A large ratio in the values of γ_0 at two adjacent joints results in *cusps*. Methods to analyze an arbitrary degree rational curve for cusps are given in (Manocha and Canny 1990). A modest reduction in the ratios of adjacent γ_0's ameliorates this effect (as shown in Fig. VI). This behavior seems to be independent of the control vertices and the weights associated with the knots. The choice of the functions $\alpha_{0_i}(u)$ is responsible for this behavior. γ_1 behaves like the *tension* parameter, and the effect is local, as shown in Fig. VII. Changes in the values of the γ_2's do not affect the position of the joints of the resulting curve (as has been shown in Fig. VIII). However, large values of the γ_2's introduce loops in a curve segment and intersections among different segments.
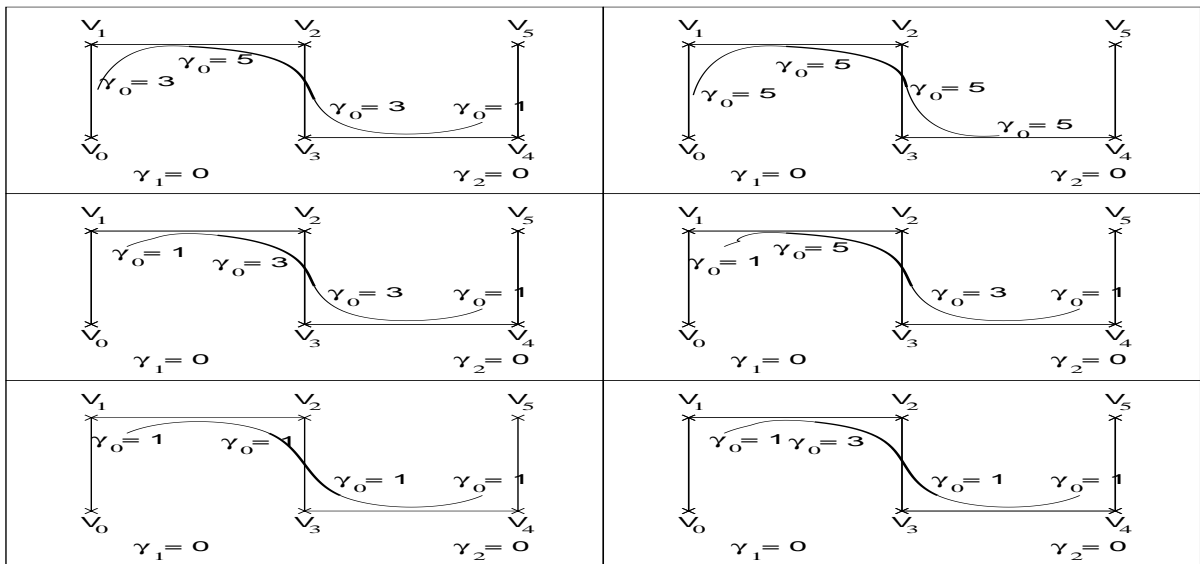

Fig. VI
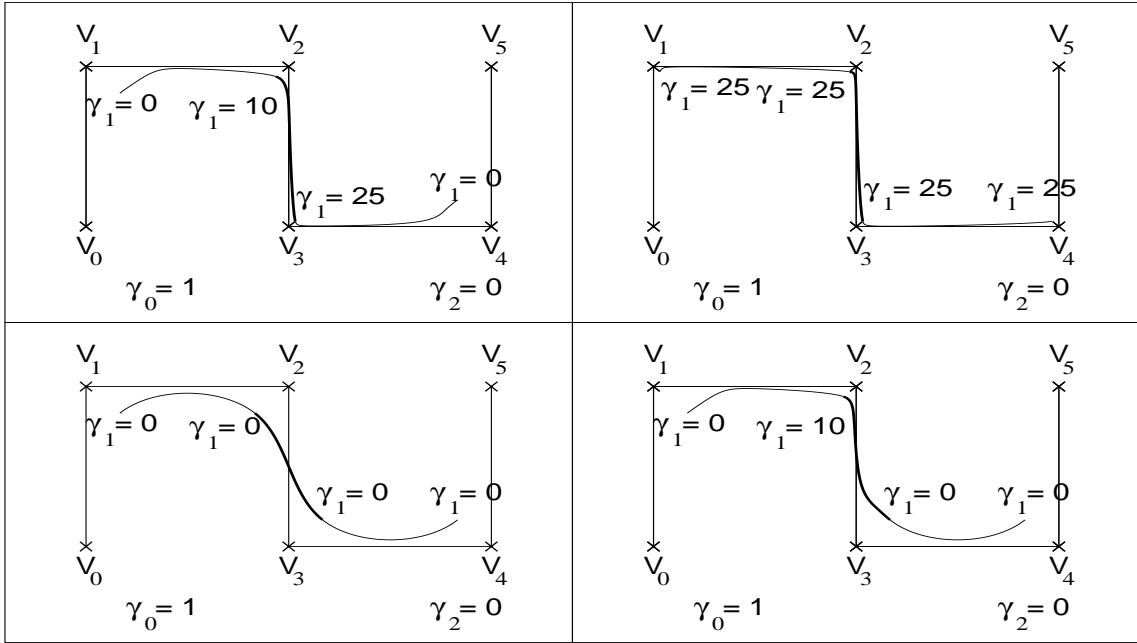The shape of the rational curves obtained by varying the γ_0's.

Fig. VII
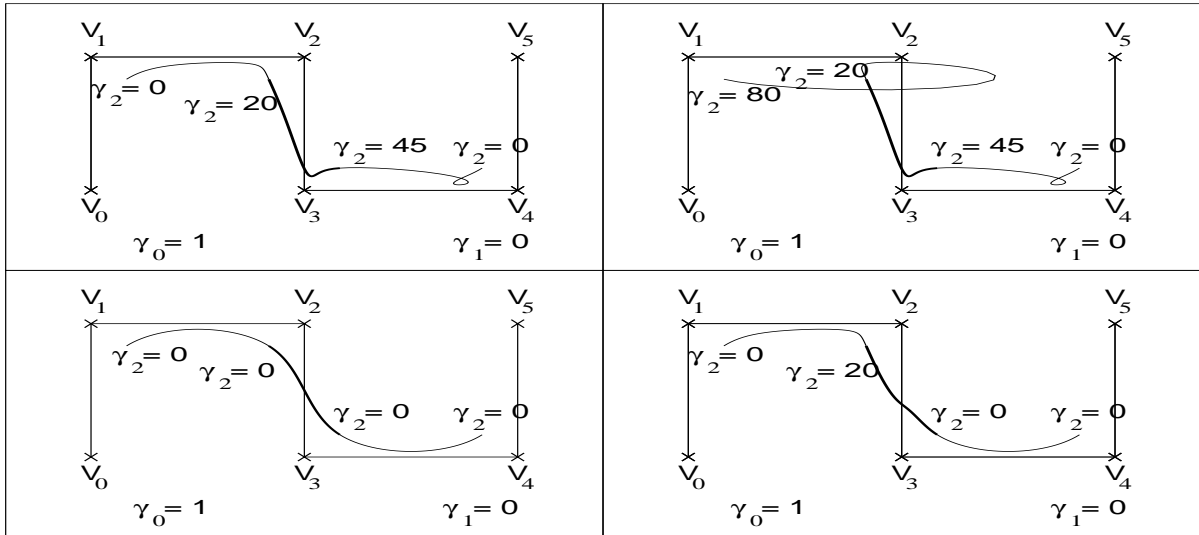
Different rational curves obtained by varying the γ_1's.

Fig. VIII

The shape of the rational curves obtained by varying the γ_2's.

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6   Discretely-Shaped Homogeneous Basis

In the previous section, we specified a *basis* for the homogeneous representation, based on the continuous variation of shape parameters. Although, it provides us with a great amount of local control, in terms of shape parameters, the evaluation cost is high. Moreover, our current formulation has undesirable features like cusps and self-intersections. We think that any function which would address these problems would be of higher degree and therefore, the evaluation cost could be more. We now introduce another basis formulation for the homogeneous representation of the rational curve, which provides a balance between the evaluation cost and the desired local control in terms of shape parameters. A *discretely-shaped* homogeneous basis is obtained by generalizing the uniform shape parameters, so that at each knot value we allow the user to specify a distinct value of each shape parameter. To be able to evaluate efficiently, we express our basis functions in terms of divided differences of one-sided power functions. Again we form a cubic basis and the resulting curve is C^2 continuous. The argument can be easily generalized to any arbitrary degree.

6.1   A Truncated Power Basis for the Homogeneous Representation

We make use of one-sided power functions and divided differences for obtaining our basis. The notation and the terminology is similar to that used in (Bartels et. al. 1987, Chapters 5,6,7). At any given knot value, the basis functions need to satisfy (3). Thus, we need a function that undergoes a jump in its zeroth, first, and second derivatives as it crosses a knot. It can be represented by a function of the type:

$$p(\overline{u}) + a_{i,i+1}(\overline{u} - \overline{u}_{i+1})_+^0 + b_{i,i+1}(\overline{u} - \overline{u}_{i+1})_+^1 + c_{i,i+1}(\overline{u} - \overline{u}_{i+1})_+^2 \tag{11}$$

The zeroth, first, and second derivatives from the left at $\overline{u} = \overline{u}_{i+1}$ are $p(\overline{u}_{i+1})$, $p^{(1)}(\overline{u}_{i+1})$, and $p^{(2)}(\overline{u}_{i+1})$, respectively. The corresponding derivatives from the right are $p(\overline{u}_{i+1}) + a_{i,i+1}$, $p^{(1)}(\overline{u}_{i+1}) + b_{i,i+1}$, and $p^{(2)}(\overline{u}_{i+1}) + 2c_{i,i+1}$, respectively. Thus, there is a jump of $a_{i,i+1}$, $b_{i,i+1}$, and $2c_{i,i+1}$ in the zeroth, first, and second derivatives, respectively, at $\overline{u} = \overline{u}_{i+1}$. Let $\alpha_{0,j}$, $\alpha_{1,j}$, and $\alpha_{2,j}$ be the values of shape parameters at the knot $\overline{u}_j$. Since the function must satisfy the constraints mentioned in (3), we have

$$
\begin{aligned}
p(\overline{u}_{i+1}) + a_{i,i+1} &= \alpha_{0,i+1}p(\overline{u}_{i+1}) \\
p^{(1)}(\overline{u}_{i+1}) + b_{i,i+1} &= \alpha_{1,i+1}p(\overline{u}_{i+1}) + \alpha_{0,i+1}p^{(1)}(\overline{u}_{i+1}) \\
p^{(2)}(\overline{u}_{i+1}) + 2c_{i,i+1} &= \alpha_{2,i+1}p(\overline{u}_{i+1}) + 2\alpha_{1,i+1}p^{(1)}(\overline{u}_{i+1}) + \alpha_{0,i+1}p^{(2)}(\overline{u}_{i+1})
\end{aligned}
\tag{12}
$$

From equations (12), the coefficients $a_{i,i+1}$, $b_{i,i+1}$, and $c_{i,i+1}$ in equation (11) can be determined. These equations show how to modify the coefficients of the functions used for representing the homogeneous basis of the *discretely-shaped* rational curve, so that they

satisfy the necessary and sufficient conditions for rational parametric C^2 continuity. To construct a one-sided basis function for the homogeneous representation, we start with $(\overline{u} - \overline{u}_i)^3_+$ since it introduces the third derivative discontinuity at $\overline{u} = \overline{u}_i$, and the resulting rational curve can be at most C^2 continuous. We modify it as we cross a knot value. Consider the following function:

$$
\begin{aligned}
h_i(\overline{u}) \quad = \quad & (\overline{u} - \overline{u}_i)^3_+ + a_{i,i+1}(\overline{u} - \overline{u}_{i+1})^0_+ + \ldots + a_{i,m+3}(\overline{u} - \overline{u}_{m+3})^0_+ + \\
& b_{i,i+1}(\overline{u} - \overline{u}_{i+1})^1_+ + \ldots + b_{i,m+3}(\overline{u} - \overline{u}_{m+3})^1_+ + \\
& c_{i,i+1}(\overline{u} - \overline{u}_{i+1})^2_+ + \ldots + c_{i,m+3}(\overline{u} - \overline{u}_{m+3})^2_+
\end{aligned}
\tag{13}
$$

We want the coefficients of $h_i(\overline{u})$ to satisfy (3). In (Manocha and Barsky 1990), we impose the constraints in the following algorithm to compute the coefficients:

$$
\begin{aligned}
&\textbf{for } \text{ i } \leftarrow \text{ 0 } \textbf{ step } \text{ 1 } \textbf{ until } \text{ m+2 } \textbf{ do} \\
&\quad Sa \;\; \leftarrow \;\; 0 \\
&\quad Sb \;\; \leftarrow \;\; 0 \\
&\quad Sc \;\; \leftarrow \;\; 0 \\
&\quad \textbf{for } \text{ j } \leftarrow \text{ i+1 } \textbf{ step } \text{ 1 } \textbf{ until } \text{ min(i+4,m+3) } \textbf{ do} \\
&\qquad p^{(0)}_{left} \leftarrow (\overline{u}_j - \overline{u}_i)^3 + Sa + \sum_{k=i+1}^{j}(b_{i,k}(\overline{u}_j - \overline{u}_k) + c_{i,k}(\overline{u}_j - \overline{u}_k)^2) \\
&\qquad p^{(1)}_{left} \leftarrow 3(\overline{u}_j - \overline{u}_i)^2 + Sb + 2\sum_{k=i+1}^{j}(c_{i,k}(\overline{u}_j - \overline{u}_k)) \\
&\qquad p^{(2)}_{left} \leftarrow 6(\overline{u}_j - \overline{u}_i) + 2Sc \\
&\qquad a_{i,j} \leftarrow (\alpha_{0,j} - 1)p^{(0)}_{left} \\
&\qquad b_{i,j} \leftarrow \alpha_{1,j}p^{(0)}_{left} + (\alpha_{0,j} - 1)p^{(1)}_{left} \\
&\qquad c_{i,j} \leftarrow \tfrac{1}{2}[\alpha_{2,j}p^{(0)}_{left} + 2\alpha_{1,j}p^{(1)}_{left} + (\alpha_{0,j} - 1)p^{(2)}_{left}] \\
&\qquad Sa \leftarrow Sa + a_{i,j} \\
&\qquad Sb \leftarrow Sb + b_{i,j} \\
&\qquad Sc \leftarrow Sc + c_{i,j} \\
&\quad \textbf{endfor} \\
&\textbf{endfor}
\end{aligned}
$$

Thus, given a knot sequence and the shape parameters at the knot values, the functions $h_i(\overline{u})$ form a basis for the homogeneous representation of C^2 rational parametric curves. We plan to use this procedure for computing the a_i's, b_i's and c_i's for the local basis defined in the next section. The *min* operator in line 5 has been specified for that purpose. The argument is analogous to that given in (Bartels et. al. 1987) for C^2 and G^2 polynomial splines.

6.2   A Local Basis for the Homogeneous Representation

We can construct any homogeneous representation for a C^2 rational curve as a linear combination of one-sided power functions. We are not allowed to choose any arbitrary linear combination. The coefficients used are determined by the shape parameters at the knot

values as shown in (12) and the resulting functions are of the type $h_i(\overline{u})$. However, these functions are computationally unsatisfying. They are only useful for their simplicity and ease of understanding. For constructing splines, they suffer from two severe shortcomings: *numerical instability* and *lack of local control* (Bartels et. al. 1987; Barsky 1988b). Therefore, we need to apply differencing operations so as to obtain a local basis and to be able to compute them quickly.

Let $\overline{u}_j \le \overline{u} < \overline{u}_{j+1}$. Equation (13) can be expressed as

$$h_i(\overline{u}) = \overline{u}^3 + [\sum_{k=i+1}^{j} c_{i,k} - 3\overline{u}_i]\overline{u}^2 + [\sum_{k=i+1}^{j} b_{i,k} - 2\sum_{k=i+1}^{j} c_{i,k}\overline{u}_k + 3\overline{u}_i^2]\overline{u} + \qquad (14)$$

$$[\sum_{k=i+1}^{j} (a_{i,k} - b_{i,k}\overline{u}_k + c_{i,k}\overline{u}_k^2) - \overline{u}_i^3]$$

Our aim is to apply some form of differencing operator to $h_i(u)$ so as to obtain a local basis. We do so by successively eliminating the higher powers of $\overline{u}$ and normalizing each time, as explained in (Manocha and Barsky 1990).

We now define the constants, which are used for normalization. Let

$$A_{i,j} = \sum_{k=j+1}^{i+4} c_{j,k} - 3\overline{u}_j$$

$$B_{i,j} = \sum_{k=j+1}^{i+4} [b_{j,k} - 2c_{j,k}\overline{u}_k] + 3\overline{u}_j^2 \qquad (15)$$

$$C_{i,j} = \sum_{k=j+1}^{i+4} [a_{j,k} - b_{j,k}\overline{u}_k + c_{j,k}\overline{u}_k^2] - \overline{u}_j^3$$

for $j = i, i+1, i+2, i+3$ and

$$A_{i,i+4} = -3\overline{u}_{i+4}$$
$$B_{i,i+4} = 3\overline{u}_{i+4}^2$$
$$C_{i,i+4} = -\overline{u}_{i+4}^3$$
$$D_{i,j} = \frac{B_{i,j+1} - B_{i,j}}{A_{i,j+1} - A_{i,j}}$$
$$E_{i,j} = \frac{C_{i,j+1} - C_{i,j}}{A_{i,j+1} - A_{i,j}}$$
$$F_{i,j} = \frac{E_{i,j+1} - E_{i,j}}{D_{i,j+1} - D_{i,j}}$$

Furthermore, we define

$$h_j(\overline{u}) = \overline{u}^3 + A_{i,j}\overline{u}^2 + B_{i,j}\overline{u} + C_{i,j} \quad for \qquad \overline{u}_{i+4} \le \overline{u} < \overline{u}_{i+5},$$

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$$\Delta_i^1 h_j(\overline{u}) \;=\; \frac{h_{j+1}(\overline{u}) - h_j(\overline{u})}{A_{i,j+1} - A_{i,j}} \qquad for\ \overline{u}_{i+4} \le \overline{u} < \overline{u}_{i+5},$$

$$\Delta_i^2 h_j(\overline{u}) \;=\; \frac{\Delta_i^1 h_{j+1}(\overline{u}) - \Delta_i^1 h_j(\overline{u})}{D_{i,j+1} - D_{i,j}} \qquad for\ \overline{u}_{i+4} \le \overline{u} < \overline{u}_{i+5},$$

$$\Delta_i^3 h_j(\overline{u}) \;=\; \frac{\Delta_i^2 h_{j+1}(\overline{u}) - \Delta_i^2 h_j(\overline{u})}{F_{i,j+1} - F_{i,j}} \qquad for\ \overline{u}_{i+4} \le \overline{u} < \overline{u}_{i+5},$$

$$\Delta_i^4 h_j(\overline{u}) \;=\; -[\Delta_i^3 h_{j+1}(\overline{u}) - \Delta_i^3 h_j(\overline{u})] \quad for\ \overline{u}_{i+4} \le \overline{u} < \overline{u}_{i+5}.$$

The function $\Delta_i^4 h_i(\overline{u})$ is defined for any value of $\overline{u}$, but we ensured that it would be zero whenever $\overline{u} \in [\overline{u}_{i+4}, \overline{u}_{i+5})$ or $\overline{u} < \overline{u}_i$. To arrange for locality, we define our homogeneous basis function $H_i(\overline{u})$ as

$$H_i(\overline{u}) = \begin{cases} 0 & \overline{u} < \overline{u}_i \ or \ \overline{u} \ge \overline{u}_{i+4} \\ \Delta_i^4 h_i(\overline{u}) & \overline{u}_i \le \overline{u} < \overline{u}_{i+4} \end{cases}$$

Given a knot sequence comprising $m + 5$ knots $\overline{u}_0, \ldots, \overline{u}_{m+4}$ and shape parameters $\alpha_{0,i}, \alpha_{1,i}$ and $\alpha_{2,i}$ associated with each knot value, we construct the $m + 1$ basis functions $H_i(\overline{u})$ in the manner defined above. Given $m + 1$ control vertices $\mathbf{V}_0, \ldots, \mathbf{V}_m$, a C^2 discretely-shaped rational curve is defined as

$$\mathbf{Q}(\overline{u}) = \sum_{i=0}^{m} \mathbf{W}_i H_i(\overline{u}) \quad \overline{u}_3 \le \overline{u} < \overline{u}_{m+1}$$

in the homogeneous space or as

$$\mathbf{q}(\overline{u}) = \sum_{i=0}^{m} \mathbf{V}_i \frac{w_i H_i(\overline{u})}{\sum_{j=0}^{m} w_j H_j(\overline{u})} \quad \overline{u}_3 \le \overline{u} < \overline{u}_{m+1}$$

in the projected space, where $\mathbf{W}_i = (w_i \mathbf{V}_i, w_i)$.

7 Rational Geometric Continuity

All our analysis thus far has been for *parametric continuity* of rational curves. For each notion of geometric continuity of parametric curves, the necessary and sufficient conditions for the corresponding geometric continuity of rational curves are derived in (Hohmeyer and Barsky 1989). Let us consider the notion corresponding to reparametrization. The necessary and sufficient conditions can be expressed as replacing the matrix consisting of $\alpha's$ in (2) by a matrix that is expressed as the product of an α matrix (the same as the one defined in (2)) and a β matrix as defined in (Barsky and DeRose 1989; Barsky and DeRose 1990; DeRose 1985). Thus, for second order rational geometric continuity, we will have five shape parameters α_0, α_1, α_2, β_1, and β_2. A rational curve $\mathbf{q}(u)$ is said to possess second

order rational geometric continuity at a knot value $u = u_0$ if and only if each component of its homogeneous representation, $\mathbf{Q}(u)$, say $f(u)$, satisfies the following relation:

$$
\lim_{u \to u_0+} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & 2\alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & \beta_2 & \beta_1{}^2 \end{bmatrix} \lim_{u \to u_0-} \begin{bmatrix} f^{(0)}(u) \\ f^{(1)}(u) \\ f^{(2)}(u) \end{bmatrix}
$$

This relation can be used for developing basis functions for second order rational geometric continuous curves, whose homogeneous representation is cubic. We can use this relation to develop all the three types of basis functions; i.e., we can develop basis functions for homogeneous representation of uniformly-shaped, continuously-shaped, and discretely-shaped curves. A similar relationship has been defined for Frenet frame continuity of rational curves (Hohmeyer and Barsky 1989)

8 Conclusion

The rational curve is more versatile than its polynomial counterpart. The connection with constructions in a higher dimensional space offers extra degrees of freedom. These degrees of freedom were exploited in (Hohmeyer and Barsky 1989) to determine the necessary and sufficient conditions for parametric and geometric continuity of rational curves. We used the exact conditions for defining the basis functions for the homogeneous representation of C^2 cubic rational curves. Although our analysis has been for parametric continuity of rational curves, we have shown how to obtain rational curves with geometric continuity as well.

We have highlighted three different formulations for constructing basis functions of C^2 continuous rational curves. Each form has its relative advantages and disadvantages in terms of evaluation cost and the degree of local control relative to the shape parameters. The shape parameters have been shown to provide us with an intuitive way of changing the curve independent of the control vertices (Manocha and Barsky 1990). In particular, we observed that two of the three uniform shape parameters, α_0 and α_1, behave like the *bias* and *tension* parameters, β_1 and β_2, respectively, of the G^2 cubic polynomial and rational Beta-splines. The third shape parameter, α_2, acts like a *joint-invariant* parameter.

The formulation of $R_i(u)$ in (4) possesses the *partition of unity* property irrespective of the $B_i(u)$'s and no normalization constraint is required on the $B_i(u)$'s. Therefore, while determining the basis functions for homogeneous representation, there are sixteen unknowns and fifteen continuity constraints (for parametric or geometric continuity). We have not exploited this property in determining basis functions for continuously-shaped or discretely-shaped curves. Moreover, this can be used as a shape parameter whenever these basis functions are used for tensor product surfaces. The use of these basis functions and shape parameters for applications involving surfaces needs further investigation.

9   Acknowledgements

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